Distinguishing Non-Standard Neutrino Interactions from Non-Unitarity

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by

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"I have done a terrible thing, I have postulated a particle that cannot be detected."

- Wolfgang Pauli, 1931.

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Abstract

The current theoretical framework of neutrino physics is reviewed, along with an explanation of the different mass models used for neutrinos and the experimental evidence for neutrino oscillations. New physics effects manifested through Non-Standard Neutrino Interactions (NSNI) and the Non-Unitarity of the light neutrino mixing matrix, due to Non-Decoupling of Heavy Neutrinos (NDHN) are also introduced. Possible ways to distinguish between NSNI and NDHN in neutrino oscillation experiments are then proposed.

Declaration

The work in this dissertation is based on research carried out at the Institute for Particle Physics Phenomenology, Durham, England. No part of this dissertation has been submitted elsewhere for any other degree or qualification. No originality is claimed for the material included in this work.

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Introduction

Neutrino Physics has attracted a great deal of attention from particle physicists in the last few years, largely due to experiments such as Superkamiokande in Japan (1997) and the Sudbury Neutrino Observatory in Canada (1999). In fact, it is fair to say that in the last few years, this area of research has become one of the fastest growing topics studied in phenomenological particle physics.

To understand where neutrino physics is placed within phenomenological particle physics, it is necessary to take a look at its progression from the early twentieth century experiments on nuclear beta decay. In beta decay an electron is ejected from the nucleus of an atom and a neutron is turned into a proton. Experimental physicists such as James Chadwick found that the electron energies observed from the decay of neutrons could not be explained using the mass difference between the neutron and the proton. The electrons should all have had a fixed energy, but instead they observed a continuous spectrum of energy. It was not until 1931 that a solution was proposed, when Wolfgang Pauli introduced a new neutral particle [1] to "save" the principle of energy conservation. But he noted from the experimental data that in order for this new particle to exist, it would have to weigh less than 1% of the proton mass and subsequently foresaw many problems involved with its detection. Shortly after this, in 1934, Enrico Fermi published his mathematical theory for beta decay [2], where he dubbed the new particle the "neutrino" and introduced the four-Fermi hamiltonian for the interaction. Fermi's new theory suggested that all the particles interacted very weakly with each other and therefore required a much smaller force than that of the only other known force relevant to nuclear and atomic physics, the electromagnetic force. This new "weak" force opened up a whole new area of theoretical physics concerned with its interactions.

However it was not until the 1950's that the first neutrino was actually discovered. Frederick Reines and Clyde Cowan [3] observed the reaction

$$\bar{\nu}_e + p \to n + e^- \ ,$$

in their famous nobel prize winning experiment, using a high intensity flux ($E_\nu \sim 1$ MeV) of electron anti-neutrinos from a nuclear reactor. Other experiments followed and in 1958 Goldhaber et al [4] measured the helicity of the neutrino, where it was determined that only left-handed neutrinos (spin anti-parallel to the direction of motion) participated in the weak interaction. Since then no right-handed neutrinos have ever been observed in experiments to provide evidence for the contrary.

In 1989, thirty years after the first neutrino was discovered, studies of the lifetime of the Z boson, the force carrier of the weak interaction to which neutrinos couple, were undertaken at the LEP collider in CERN. It was determined that there were only three light 'active' neutrino species present in nature, all with masses much less than 45 GeV. However neutrinos with even weaker couplings to the Z boson, known as "sterile" neutrinos were not completely ruled out. In fact the question of whether any sterile neutrinos exist is now crucial in understanding all of the experimental evidence gathered so far. It seems sterile neutrinos may actually be needed to match up our current theories of neutrino physics with the data taken from experiments.

The most important question that is still unanswered about neutrinos is that of whether they have mass or not and if so, what type of mass model should be used to describe the mass. This problem of neutrino mass has huge implications not only within particle physics areas such as CP violation, but also in cosmological theories concerned with the evolution of the universe. In the past decades many detectors were built to test neutrino theoretical models such as that of the Standard Solar Model, which describes the energy spectrum of neutrinos from the sun. These detectors were not necessarily built to determine whether or not neutrinos have mass, but they have observed deficits in certain neutrino species coming from the sun, atmosphere and reactors. It was suggested in 1958 by Bruno Pontecorvo [5], that if neutrinos had mass then it would be possible for them to oscillate into each other. Consequently it is thought that these detectors measuring deficits in certain neutrino species are actually measuring neutrino oscillations as a result of neutrino mass.

All theoretical models concerning our present knowledge of particle physics which have been experimentally confirmed, are contained within the Standard Model. Since its completion in the 1970's it has been very successful in describing data from every particle physics experiment except for those from certain neutrino experiments. However it is not thought that the standard model gives a complete picture of nature, as there are many aspects other than the data from neutrino experiments that cannot be explained within its framework. Aspects such as the number of particle species or their masses are still unexplained and several extensions have been proposed as an answer, such as Supersymmetric and Grand Unified Theories. But the energy scale of these theories is $\sim \mathcal{O}(10^{16})$ GeV and so it will be a long time before they can actually be tested in conventional particle physics experiments.

Neutrino physics may provide an invaluable way to probe this physics beyond the standard model, known as “New Physics”. It is the aim of this dissertation to provide an overview of the theory behind neutrinos and new physics effects manifested through Non-Standard Neutrino Interactions and the Non-Decoupling of Heavy Neutrinos. These new physics effects are very strongly linked to the neutrino mass problem and may be detectable much sooner at neutrino “oscillation” experiments than through conventional particle physics experiments. It is therefore necessary for us to have a phenomenological framework to be able to distinguish between the different new physics effects that will be observed.

M. S. Tame
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Chapter 1

Theoretical Framework of Neutrino Physics

1.1 Neutrinos in the Standard Model

The Standard Model (SM) is a description of three out of the four known forces in nature, these are the strong, weak and electromagnetic forces. Each force is mediated by a spin-1 particle (gauge boson) and the relation between these particles to local (gauge) symmetries is the basis for the description of interactions within the SM. The strong, weak and electromagnetic interactions are related to the gauge symmetry groups $SU(3)$, $SU(2)$ and $U(1)$ respectfully. The rest of the elementary particles that do not carry a force can be split up into two types known as leptons and quarks, all of which have spin-1/2 (fermions). These fermions are then affected by the different types of interactions with the gauge bosons depending on their representations under the corresponding gauge symmetry group. From the incorporation of all the relevant gauge symmetry groups involved in the interactions, the gauge group of the SM is $SU(3)_C \times SU(2)_L \times U(1)_Y$.

Each quark comes in three different colours which results in the $SU(3)_C$ gauge group responsible for the strong interactions. There are many books that provide a detailed description of Quantum Chromodynamics (QCD) concerning this gauge group [6], but since neutrinos are leptons and are not affected by the strong force this symmetry group will not be dealt with here.

The Glashow-Weinberg-Salam (GWS) theory [7] also known as the Electroweak (EW) Model unifies the electromagnetic and weak interactions and results in the enlarged $SU(2)_L \times U(1)_Y$ gauge group. However this is not so much a unification than a mixing of gauge groups. The $SU(2)_L$ left-handed weak isospin group associated with three gauge bosons $A_\mu^{1,2,3}$ and the weak hypercharge group $U(1)_Y$ associated with one gauge boson a_μ are mixed. The field B_μ for the photon, the gauge boson that mediates the electromagnetic force, becomes a linear combination of the four gauge bosons from the enlarged gauge group

$$B_\mu = \cos(\theta_W) a_\mu + \sin(\theta_W) A^3_\mu \ , \tag{1.1}$$

where θ_W is the Weinberg angle and is given by the $SU(2)_L$ coupling constant g and the $U(1)_Y$ coupling constant g' in the relation

$$\cos(\theta_W) = \frac{g^2}{\sqrt{g^2 + g'^2}} \qquad \text{and electromagnetic charge } e = g\sin(\theta_W) \ . \tag{1.2}$$

The fields for the charged gauge bosons W^+, W^- and the neutral gauge boson Z that mediate the weak force are given by

$$Z_\mu = \sin(\theta_W) a_\mu - \cos(\theta_W) A^3_\mu \ , \tag{1.3}$$

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \mp A^2_\mu) \ . \tag{1.4}$$

The electromagnetic charge Q of a fermion can then be expressed in terms of the third component of weak isospin of its left chiral projection I_3 and its weak hypercharge Y

$$Q = \frac{Y}{2} + I_3 \ . \tag{1.5}$$

There are three different sectors within the SM; the gauge sector, the fermionic sector and the Higgs sector. These sectors couple to themselves and to each other in certain ways described by the lagrangian of the SM: $\mathcal{L}_{SM}$, which defines all the physics of this model.

The fermions in the SM are separated into leptons and quarks, where they can be described by three generations of doublets corresponding to their transformations under the $SU(2)_L$ gauge group

$$\begin{array}{cccc} 1^{st} & 2^{nd} & 3^{rd} & \\ \begin{pmatrix} u \\ d \end{pmatrix}_{\rm L} & \begin{pmatrix} c \\ s \end{pmatrix}_{\rm L} & \begin{pmatrix} t \\ b \end{pmatrix}_{\rm L} & \text{Quarks} \\ \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_{\rm L} & \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_{\rm L} & \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_{\rm L} & \text{Leptons} \end{array} \tag{1.6}$$

For simplicity one can write

$$L^\ell_L = \begin{pmatrix} \nu_{\ell L} \\ \ell^-_L \end{pmatrix} \ , \qquad \ell = \{e, \mu, \tau\} \ . \tag{1.7}$$

Where e, μ, τ are the charged lepton mass fields[1] and ν_e, ν_μ, ν_τ are the neutrino weak interaction fields. The use of weak interaction fields for the neutrinos

[1] The fields that correspond to a diagonalised charged lepton mass matrix.

instead of their mass fields will be discussed later on. For the quark doublets one can write

$$Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix} \ , \quad u^i = \{u, c, t\} \quad d^i = \{d, s, b\} \ . \tag{1.8}$$

Right-handed terms can also be introduced for all the fermions, but in the form of singlets under a $SU(2)_L$ gauge transformation. Therefore not **all** are involved in electroweak interactions. Only those representations that carry a $SU(2)_L \times U(1)_Y$ quantum number (Q, I_3 or Y) can interact. For the leptons these are

$$E_R^\ell = \{e_R, \mu_R, \tau_R\} \ , \tag{1.9}$$

and for the quarks these are

$$U_R^i = \{u_R, c_R, t_R\} \ , \qquad D_R^i = \{d_R, s_R, b_R\} \ . \tag{1.10}$$

Therefore there are five different representations overall of the SM gauge group for each generation of particle $i, \ell = \{1, 2, 3\}$

$$Q_L^i(3,2)_{+1/6} \ , \ U_R^i(3,1)_{+2/3} \ , \ D_R^i(3,1)_{-1/3} \ , \ L_L^\ell(1,2)_{-1/2} \ , \ E_R^\ell(1,1)_{-1} \ . \tag{1.11}$$

Here the first number in the bracket signifies how the representation transforms under a $SU(3)_C$ gauge transformation. The second corresponds to a $SU(2)_L$ gauge transformation and the subscript corresponds to the hypercharge under the $U(1)_Y$ gauge transformation (1-singlet, 2-doublet, 3-triplet).

Right-handed representations of neutrinos cannot be inferred from electroweak interactions as they carry no $SU(2)_L \times U(1)_Y$ quantum number. For instance a right-handed representation of a neutrino would have a zero I_3 number as it transforms as a singlet under a $SU(2)_L$ gauge transformation. It also has no charge Q and therefore from Eq. (1.5) has no relevant quantum number. Subsequently right-handed neutrinos cannot be inferred from any interactions in the SM, as they would transform as a singlet under the entire gauge group. Neutrinos that do this are commonly known as "sterile" neutrinos when introduced into extensions of the SM. The left-handed neutrinos involved in the weak interaction are known as "active" neutrinos as they do transform under the SM gauge group. Right-handed neutrinos have not been detected in colliders at the present energies attainable and the SM which is used to interpret these results is assumed to be part of some larger symmetry group which includes right-handed neutrinos at a higher energy scale. Therefore if right-handed neutrinos exist in nature it is most likely that their mass is far above the energy currently available to us. But it is also possible that they do not exist at all. If right-handed neutrinos do exist, then their influence might be seen through neutrino mass models and their surrounding theory, even at the energies in present experiments.

However before discussing neutrino mass, it is first necessary to review the weak interactions of left-handed neutrinos. Here the contribution of the electroweak theory to the SM lagrangian $\mathcal{L}_{SM}$ involves terms that couple the left-handed neutrinos to the Z and $W^{\pm}$ gauge bosons, but not to the photon, as neutrinos have no electric charge. The coupling to the Z boson is of the form

$$\mathcal{L}_{Z\nu\bar{\nu}} = -\frac{g}{2\cos\theta_W} \underbrace{\sum_{\ell} \bar{\nu}_\ell \gamma_\mu \frac{1}{2}(1-\gamma_5)\nu_\ell}_{J_\mu^{NC}} Z^\mu \ . \tag{1.12}$$

The axial-vector form of the neutrino neutral charged (NC) current J_μ^{NC} ensures that only left-handed neutrinos $\nu_{\ell L}$ are involved in this coupling

$$J_\mu^{NC} = \sum_\ell \bar{\nu}_{\ell L} \gamma_\mu \nu_{\ell L} \ . \tag{1.13}$$

To determine the number of neutrino species present in the SM one can make use of Eq. (1.12) and measure the Z total decay width, then provided the mass of each neutrino $m_{\nu_\ell} \ll \frac{M_Z}{2}$, they will all contribute the same amount to the Z total decay width. From Eq. (1.12) one finds the Z-width for neutrinos [8]

$$\Gamma(Z \to \nu_\ell \bar{\nu}_\ell) = \frac{\sqrt{2} G_F M_Z^3}{24\pi} \rho \ , \tag{1.14}$$

where G_F is the Fermi constant determined from μ decay [9] and ρ is related to the axial coupling of the charged leptons to the Z-boson ($\rho \simeq 1$). Using values from [8, 9], the Z-width for neutrinos becomes

$$\Gamma_\nu \equiv \Gamma(Z \to \nu_\ell \bar{\nu}_\ell) = (167.06 \pm 0.22) \text{ MeV} \ . \tag{1.15}$$

To find the number of different neutrino species N_ν, precision measurements of the Z total decay width and its partial decay into hadrons (combinations of quarks) and leptons can be used

$$\Gamma_{\text{total}} = \Gamma_{\text{hadrons}} + 3\Gamma_{\text{leptons}} + N_\nu \Gamma_\nu \ . \tag{1.16}$$

Data from [8] gives $N_\nu = 2.992 \pm 0.020$. As mentioned before these three species of neutrinos are left-handed active neutrinos as they participate in interactions within the SM. In an extension of the SM it is possible that both left- and right-handed neutrinos could be introduced as sterile.

The contribution of the electroweak theory to the SM lagrangian $\mathcal{L}_{SM}$ also involves terms that couple the left-handed neutrinos and the charge leptons to the $W^{\pm}$ gauge bosons

$$\mathcal{L}_{W\ell\nu_\ell} = \frac{g}{\sqrt{2}} \{ W_+^\mu J_{\mu-}^{\text{lept}} + W_-^\mu J_{\mu+}^{\text{lept}} \} \ , \tag{1.17}$$

where the leptonic charged currents (CC) are

$$J_{\mu-}^{\text{lept}} = (J_{\mu+}^{\text{lept}})^\dagger = \sum_\ell \bar{\ell} \gamma_\mu \frac{1}{2}(1-\gamma_5)\nu_\ell \equiv \sum_\ell \bar{\ell}_L \gamma_\mu \nu_{\ell L} \ . \tag{1.18}$$

The charged lepton fields in Eq. (1.18) are not in general mass fields. Therefore it is necessary to diagonalise the charged lepton mass matrix by a bi-unitary transformation of the left- and right-handed charged lepton fields. Appendix A gives a clear explanation as to why a bi-unitary transformation is needed. The fields become

$$\ell_{\mathrm{L}} = U^{\ell}\tilde{\ell}_{\mathrm{L}} \ , \qquad \ell_{\mathrm{R}} = V^{\ell}\tilde{\ell}_{\mathrm{R}} \ , \tag{1.19}$$

and the charged currents in Eq. (1.18) become

$$J^{\mathrm{lept}}_{\mu -} = (J^{\mathrm{lept}}_{\mu +})^{\dagger} = \sum_{\ell k} \overline{\tilde{\ell}_{L}}\gamma_{\mu}(U^{\ell})^{\dagger}_{\ell k}\nu_{kL} = \sum_{\ell} \overline{\tilde{\ell}_{L}}\gamma_{\mu}\tilde{\nu}_{\ell L} \ , \tag{1.20}$$

where the neutrino fields ν_k are seen as rotations of some random neutrino fields in another basis

$$\nu_{kL} = (U^{\ell})_{k\ell}\tilde{\nu}_{\ell \mathrm{L}} \ . \tag{1.21}$$

The fields $\tilde{\nu}_{\ell}$ in this other basis are called weak interaction fields, since they are produced in the decay of a W^{+} boson in association with a physically charged lepton $\tilde{\ell}$ of definite mass (the charged lepton mass matrix is now diagonalised). It is interesting to note that the neutral current in Eq. (1.13) is also not affected by the use of these new transformed neutrino fields and they can therefore be pair produced by the Z-boson also. The $\tilde{\ell}_L$ mass fields and $\tilde{\nu}_{\ell_L}$ weak interaction fields are those used in the doublet representations of the $SU(2)_L$ group as given in Eq. (1.7). Hence from now on, the three species of neutrinos as detected in the Z total decay width will be known as flavours, as they are associated with the flavours of the charged leptons.

1.2 Neutrino mass models

1.2.1 Dirac mass term

The charged leptons and quarks are Dirac particles as a consequence of electric charge conservation. They obey the Dirac equation and are each described by a Dirac field ψ_D. A Dirac field is a 4-component complex object which transforms as a spinor representation of the Lorentz group. Appendices B.1 and B.2 have a more detailed discussion of these representations and the two 2-component Weyl spinors that make up the Dirac spinor. It follows from Appendix B.2 that the mass term for a Dirac field ψ and its conjugate $\overline{\psi} = \psi^{\dagger}\gamma^{0}$ is of the form

$$\mathcal{L}^{D}_{\mathrm{mass}} = -m\overline{\psi}\psi = -m(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L) \ . \tag{1.22}$$

If neutrinos were also Dirac particles then this coupling term could come from the Yukawa interaction of a right-handed singlet ν^{ℓ}_{R} with a left-handed doublet $L^{\ell}_{L} = \begin{pmatrix} \nu_{\ell L} \\ \ell^{-}_{L} \end{pmatrix}$ via a Higgs doublet $\Phi = \begin{pmatrix} \phi^{0} \\ \phi^{-} \end{pmatrix}$. The interaction term would be introduced into the SM lagrangian $\mathcal{L}_{SM}$ as

$$\mathcal{L}_{\text{Yukawa}} = -\Gamma \bar{\nu}_R \Phi L_L + \text{h.c.}\ , \tag{1.23}$$

where Γ is the interaction strength. This would give a Dirac mass of $m_D = \Gamma < \phi^0 >$ where $< \phi^0 >$ is the vacuum expectation value (v.e.v) of the Higgs field ($\sim$ 180 GeV) gained from spontaneous symmetry breaking [11]. Similar terms like Eq. (1.23) can be added to the SM lagrangian $\mathcal{L}_{SM}$ for any quark or charged lepton, however for a small neutrino mass this would require a tiny interaction strength $\Gamma \sim 10^{-11}$.

The mass term in Eq. (1.22) is the only possible allowed mass term for neutrinos within the SM due to invariance under the SM gauge transformations. In general the lagrangian of a certain theory must be invariant under any symmetries of that theory. For the SM this includes its overall gauge group and because this is a relativistic theory, it also includes Lorentz invariance. For example, a transformation of the mass term in Eq. (1.22) above under a local $U(1)$ gauge group, such as that corresponding to the electromagnetic charge is invariant and thus the corresponding quantum number Q is conserved

$$\psi \rightarrow e^{i\alpha(x)}\psi\ ; \qquad \bar{\psi} \rightarrow e^{-i\alpha(x)}\bar{\psi}\ . \tag{1.24}$$

There are many other symmetry groups that need to be taken into consideration. There are accidental global symmetries that arise at the perturbative level in the SM lagrangian $\mathcal{L}_{SM}$ as a result of renormalisation. So in addition to the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, there is the following accidental global symmetry

$$G_{SM}^{\text{global}} = U(1)_B \times U(1)_{L_e} \times U(1)_{L_\mu} \times U(1)_{L_\tau}\ , \tag{1.25}$$

where $U(1)_B$ is the baryon number symmetry and $U(1)_{L_{\{e,\ \mu,\ \tau\}}}$ is the respective lepton number symmetry. Therefore Eq. (1.22) must also be invariant under the following $U(1)$ transformations

$$\psi_q \rightarrow e^{i\theta_B}\psi_q\ ; \qquad \psi_\ell \rightarrow e^{i\theta_L}\psi_\ell\ , \tag{1.26}$$

where ψ_q are the quark fields and ψ_ℓ are the lepton fields. One finds that it is is indeed invariant, however baryon and lepton number turn out to be anomalous. This means that the symmetry of the respective fields' classical lagrangian gets broken by quantum corrections and the transformations of Eq. (1.26) are no longer valid. There is a non-anomalous sub-group of G_{SM}^{global} called $U(1)_{B-L}$ which is found to be a good symmetry in the full quantum theory of the SM. So in general for particles described by the SM carrying any $U(1)$ quantum number, Eq. (1.22) is the only possible mass term as it inherently preserves these quantum numbers.

The overall contribution to $\mathcal{L}_{SM}$ for three generations of Dirac neutrinos using Eq. (1.22) would be

$$\mathcal{L}^{D}_{\text{mass}} = -\sum_{s,k=1}^{3}\left(\overline{\nu_{sR}}(m_D)_{sk}\nu_{kL} + \overline{\nu_{kL}}(m_D)^{\dagger}_{ks}\nu_{sR}\right) \ . \tag{1.27}$$

However as mentioned previously these fields would not be the mass fields. Nor would they even be the weak interaction fields, as they would have to be rotated by the charged lepton unitary matrix $(U^{\ell})^{\dagger}$ as in Eq. (1.20) and rotated by the unitary matrix U^{ν} associated with diagonalising the neutrino Dirac mass matrix m_D in Eq. (1.22) above.

The Dirac mass matrix can be diagonalised by the unitary matrices U^{ν} and V^{ν} as follows

$$m_D = V(m_D)^{diag}U^{\dagger} \ , \tag{1.28}$$

where $(m_D)^{diag}$ is a positive definite diagonal matrix $(m_D)^{diag}_{im} = m_i\delta_{im}$ and $m_i \geq 0$. This allows the Dirac mass term to then be written in the diagonal form

$$\mathcal{L}^{D}_{\text{mass}} = -\sum_{i=1}^{3} m_i\bar{\nu}_i\nu_i \ , \tag{1.29}$$

with

$$\nu_{kL} = \sum_{i=1}^{3} U^{\nu}_{ki}\nu_{iL} \ , \tag{1.30}$$

and

$$\nu_{sR} = \sum_{i=1}^{3} V^{\nu}_{si}\nu_{iR} \ . \tag{1.31}$$

The fields in Eq. (1.30) also need to be rotated by the charged lepton unitary matrix U^{ℓ}

$$J^{\text{lept}}_{\mu-} = (J^{\text{lept}}_{\mu+})^{\dagger} = \sum_{\ell,k=1}^{3}\overline{\tilde{\ell}}_L\gamma_{\mu}(U^{\ell})^{\dagger}_{\ell k}\nu_{kL} = \sum_{\ell,k,i=1}^{3}\overline{\tilde{\ell}}_L\gamma_{\mu}\underbrace{(U^{\ell})^{\dagger}_{\ell k}U^{\nu}_{ki}\nu_{iL}}_{\tilde{\nu}_{\ell L}} \ , \tag{1.32}$$

where the weak interaction fields

$$\tilde{\nu}_{\ell L} \xrightarrow{\text{drop tilde}} \nu_{\ell L} = \sum_{k,i=1}^{3}\underbrace{(U^{\ell})^{\dagger}_{\ell k}U^{\nu}_{ki}}_{U_{\text{mix}}}\nu_{iL} \ . \tag{1.33}$$

It is tempting to think that if neutrinos have mass then they will be Dirac particles. But as no right-handed neutrinos are allowed within the SM, a mass term in the form of Eq. (1.22) cannot be made. If a Dirac mass term is to be introduced for the neutrinos one must look beyond the SM gauge group, so as to include right-handed neutrino fields. However if this is going to be considered as an option, then it also seems reasonable to enquire about other possible mass terms. Mass terms that may not necessarily conserve the global gauge symmetries of the SM, but will be lorentz invariant. The framework of the SM does not have to be adhered to anymore.

1.2.2 Majorana mass term

Neutrinos do not carry an electromagnetic charge and therefore do not need to conserve the $U(1)_{EM}$ quantum number Q. If one gets rid of all such $U(1)$ symmetries thereby breaking the $U(1)_{B-L}$ symmetry in the process, then a new mass term can be introduced using Majorana fields. These fields do not conserve any additive quantum number, including baryon and lepton number, as they are their own antiparticle eg. one cannot assign $L = +1$ to a Majorana particle and $L = -1$ to its antiparticle. These particles therefore violate $B - L$ symmetry by two units. Appendix B.3 provides a clear introduction to Majorana spinors, where it is shown that a Majorana mass term for neutrinos would take the form

$$\mathcal{L}^M_{\rm mass} = -\frac{1}{2}(\overline{\nu_R}\tilde{C}m_S\overline{\nu_R}^T + \nu_R^T\tilde{C}m_S^\dagger\nu_R) - \frac{1}{2}(\nu_L^T\tilde{C}m_T\nu_L + \overline{\nu_L}\tilde{C}m_T^\dagger\overline{\nu_L}^T)\ . \quad (1.34)$$

The mass term above for the right-handed neutrinos is a $SU(2)_L \times U(1)_Y$ invariant, it can therefore be introduced into the SM as a bare mass term. It is possible to generate the mass term of the left-handed neutrino fields above in a similar way to that of the Dirac mass, but via a Higgs triplet field $\vec{\Delta}$. The interaction term would be introduced into the SM lagrangian $\mathcal{L}_{SM}$ as

$$\mathcal{L}_{\rm triplet} = -\frac{1}{2}\{\Gamma_T L^T\tilde{C}\vec{\tau}\cdot\vec{\Delta}L\} + \text{h.c.}\ , \quad (1.35)$$

where Γ_T is the interaction strength. When the neutral component of $\vec{\Delta}$ gained a v.e.v by spontaneous symmetry breaking, the interaction term would become

$$\mathcal{L}_{\rm mass} = -\frac{1}{2}\Gamma_T\langle\Delta^o\rangle\{\nu_L^T\tilde{C}\nu_L\} + \text{h.c.}\ . \quad (1.36)$$

This would give a Majorana mass of $m_T = \Gamma_T < \Delta^0 >$, where $< \Delta^0 >$ is the v.e.v of the neutral component Δ^0 of the Higgs triplet field.

In many extensions of the SM this Higgs triplet field cannot be introduced naturally ie. as part of a renormalisable term, but there are exceptions to this, such as the left-right symmetric models [14]. Another problem is that the Yukawa coupling for the Higgs triplet field must be unnaturally small $\Gamma_T \sim 10^{-9}$, a similar situation to that of the Dirac mass model. It would therefore be more appealing to have the mass generated by some new physics (NP). Here one must also consider non-renormalisable terms that are consistent with the gauge symmetry and fermionic content of the SM, but not necessarily the global symmetries. In doing so, the SM is treated as an effective theory of nature and terms involving NP can be introduced. A set of non-renormalisable dimension-five Yukawa interaction terms could be introduced and an effective Higgs triplet could subsequently be made out of two Higgs doublets in the form

$$\mathcal{L}^{d=5}_{\rm eff} = \frac{g}{2\Lambda_{\rm NP}}(L^T\tilde{C}\vec{\tau}L)\cdot\underbrace{(\Phi^T C\vec{\tau}\Phi)}_{\vec{\Delta}} + \text{h.c.}\ . \quad (1.37)$$

Here g is a coupling constant, $\Lambda_{\rm NP}$ is the scale of the new physics and C is the charge conjugation matrix associated with the Higgs doublets. There are also other extensions of the Higgs sector such as the Zee model [15], which would also allow for a left-handed Majorana mass term by the introduction of a singlet and two Higgs doublet fields.

None of the models mentioned above for the left-handed Majorana neutrinos shall be discussed here, suffice to say that their mass terms could be generated through several competing theories involving the Higgs field.

Concentrating on the contribution to $\mathcal{L}_{SM}$ for three generations of left-handed Majorana neutrinos one would have

$$\mathcal{L}^{M(L)}_{\rm mass} = -\frac{1}{2}\sum_{k,j=1}^{3}\left(\nu^T_{kL}\tilde{C}(m_T)_{kj}\nu_{jL} + \overline{\nu_{kL}}\tilde{C}(m_T)^\dagger_{kj}\overline{\nu_{jL}}^T\right) \ , \tag{1.38}$$

where upon diagonalisation of the mass matrix one obtains

$$\nu_{kL} = \sum_{i=1}^{3} U^\nu_{ki}\nu_{iL} \ , \tag{1.39}$$

and once again after a rotation with the charged lepton unitary matrix, one obtains the weak interaction fields

$$\nu_{\ell L} = \sum_{k,i=1}^{3} \underbrace{(U^\ell)^\dagger_{\ell k}U^\nu_{ki}}_{U_{\rm mix}}\nu_{iL} \ . \tag{1.40}$$

1.2.3 Dirac-Majorana mass term

The overall possible contribution to the SM lagrangian $\mathcal{L}_{SM}$ from both the Majorana and Dirac terms would be

$$\begin{aligned}\mathcal{L}^{D+M}_{\rm mass} = & - (\overline{\nu_R}m_D\nu_L + \overline{\nu_L}m^\dagger_D\nu_R) - \frac{1}{2}(\overline{\nu_R}\tilde{C}m_S\overline{\nu_R}^T + \nu^T_R\tilde{C}m^\dagger_S\nu_R) \\ & - \frac{1}{2}(\nu^T_L\tilde{C}m_T\nu_L + \overline{\nu_L}\tilde{C}m^\dagger_T\overline{\nu_L}^T) \ . \end{aligned} \tag{1.41}$$

If there are only three generations of neutrinos as predicted by the Z total decay width, then the mass matrices m_D, m_S and m_T would be 3×3 matrices. The neutrino fields above would then be combinations of the mass fields, the fields associated with the eigenvalues of the mass matrices after diagonalisation.

The Dirac mass term m_D conserves fermion number ($B-L$ symmetry) and as it transforms as a doublet representation of $SU(2)_L$, it would be generated after electroweak spontaneous symmetry breaking of $SU(2)_L\times U(1)_Y$. The mass term m_T transforms as a triplet representation of $SU(2)_L$ and it would also be generated after electroweak spontaneous symmetry breaking. The mass term m_S couples ν_R's with themselves and is therefore a $SU(2)_L\times U(1)_Y$ invariant. The mass terms with m_S and m_T both violate fermion number by two units as they involve Majorana particles. In summary, both m_D and m_T must originate after $SU(2)_L \times U(1)_Y$ electroweak spontaneous symmetry breaking of the SM, whereas m_S appears as a bare mass term and its scale would be assumed to be much higher than that of the symmetry breaking.

By introducing right-handed neutrinos into an extension of the SM there would be no symmetry that protects the neutrinos from becoming massive; If the $U(1)_{B-L}$ symmetry is kept as a global symmetry, then Dirac mass terms could be used and if the $U(1)_{B-L}$ symmetry is broken, then Majorana mass terms could be used.

If one assumes that both mass models might exist thereby needing a Higgs singlet, doublet and triplet, then the two models can be combined into one and interesting scenarios can be investigated. From Appendix B.3 the relationship between a charge conjugate field and a transposed field is written as

$$\psi^c = \tilde{C}\bar{\psi}^T \ , \quad \overline{\psi^c} = \psi^T \tilde{C} \ . \tag{1.42}$$

This allows the terms in Eq. (1.41) to be written in a more compact and symmetrical way. To see this the Dirac terms must be written as follows

$$\overline{\nu_{\rm R}}\nu_{\rm L} = -\nu_{\rm L}^T \overline{\nu_{\rm R}}^T = \nu_{\rm L}^T \tilde{C}\tilde{C}\overline{\nu_{\rm R}}^T = \overline{\nu_{\rm L}^c}\nu_{\rm R}^c \ , \tag{1.43}$$

giving

$$\overline{\nu_{\rm R}}\nu_{\rm L} = \frac{1}{2}[\overline{\nu_{\rm R}}\nu_{\rm L} + \overline{\nu_{\rm L}^c}\nu_{\rm R}^c] \ . \tag{1.44}$$

Subsequently one can write the overall possible contribution to the SM lagrangian from both mass models as

$$\mathcal{L}_{\rm mass} = -\frac{1}{2}\left[\underbrace{(\overline{\nu_{\rm L}^c}\ \overline{\nu_{\rm R}})}_{\bar{\chi}_L^c}\underbrace{\begin{pmatrix} m_T & m_D^T \\ m_D & m_S \end{pmatrix}}_{M_\nu}\underbrace{\begin{pmatrix} \nu_{\rm L} \\ \nu_R^c \end{pmatrix}}_{\chi_L}\right] + \text{h.c.} \ , \tag{1.45}$$

where the left-handed column vector χ_L has been introduced. Here $B - L$ symmetry is not conserved unless $m_T = m_S = 0$. This term describes one generation of neutrino, but it can be easily generalised. It is important to note that the fields above are not be the weak interaction fields, the complex symmetric mass matrix M_ν will need to be diagonalised and a rotation with the charged lepton unitary matrix U^ℓ will be required (as shown previously). If the lepton content of a particular theory is

$$L_L^\ell(1,2)_{-1/2} \ , \ E_R^k(1,1)_{-1} \ , \ \nu_R^k(1,1)_0 \qquad \ell = \{1,..,n_L\} \ , \quad k = \{1,..,n_R\} \ , \tag{1.46}$$

then m_T, m_D and m_S will be matrices of dimension $n_L \times n_L$, $n_R \times n_L$ and $n_R \times n_R$ respectively. So for three generations, M_ν would become a 6×6 matrix. If the eigenvalues of the mass matrix M_ν are to be found, then diagonalisation with a unitary matrix is required

$$(U^\nu)^\dagger M_\nu U^\nu = (M_\nu)_{\text{diag}} \ . \tag{1.47}$$

with $(M_\nu)_{\text{diag}} = m_i\delta_{ij}, \ m_i > 0$. Putting this into Eq. (1.45) one obtains

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\left[(\overline{\nu^c_{\text{L}}} \ \overline{\nu_{\text{R}}})U^\nu (M_\nu)_{\text{diag}}(U^\nu)^\dagger \begin{pmatrix} \nu_{\text{L}} \\ \nu^c_R \end{pmatrix}\right] + \text{h.c.} \ . \tag{1.48}$$

For several generations ν_L and ν_R are the column vectors of the left- and right-handed fields respectively. Introducing Majorana fields η as

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \end{pmatrix} = (U^\nu)^\dagger \underbrace{\begin{pmatrix} \nu_{\text{L}} \\ \nu^c_R \end{pmatrix}}_{\chi_L} + \left((U^\nu)^\dagger \underbrace{\begin{pmatrix} \nu_{\text{L}} \\ \nu^c_R \end{pmatrix}}_{\chi_L}\right)^c \ , \tag{1.49}$$

allows the mass term to be written as

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\bar{\eta}(M_\nu)_{\text{diag}}\eta = -\frac{1}{2}\sum_{i=1}^{n_L+n_R} m_i\bar{\eta}_i\eta_i \ , \tag{1.50}$$

and it can be seen that in the case of combined a Dirac-Majorana mass term the Majorana fields η are the fields with definite mass or physical fields.

From Eq. (1.49) one obtains the mixing relations

$$\nu_{kL} = \sum_{i=1}^{n_L+n_R} U^\nu_{ki}\eta_{iL} \ , \tag{1.51}$$

$$(\nu_{sR})^c = \sum_{i=1}^{n_L+n_R} U^\nu_{si}\eta_{iL} \ . \tag{1.52}$$

Upon rotation of the charged lepton unitary matrix U^ℓ one obtains

$$\nu_{\ell L} = \sum_{i=1}^{n_L+n_R} \underbrace{(U^\ell)^\dagger_{\ell k} U^\nu_{ki}}_{U_{\text{mix}}} \eta_{iL} \ . \tag{1.53}$$

Therefore the weak interaction fields ν_ℓ are superpositions of the left-handed components of the Majorana fields η_i of definite mass. If the masses m_i are small then Eqs. (1.51) and (1.52) imply that the weak interaction neutrinos could be superpositions of the sterile neutrinos ν_{sR}. It should be noted that this combined mass model does not limit the physical fields to being just Majorana. In the case of a Dirac neutrino field with $m_S = m_T = 0$ one can represent it as a mixture of two Majorana fields with the same mass but opposite CP parities, thus removing half the degrees of freedom

$$\nu_D = \frac{1}{\sqrt{2}}(-i\eta_1 + \eta_2) \ . \tag{1.54}$$

1.3 The See-saw mechanism

In the Dirac model of neutrinos there is no natural theoretical explanation as to the light masses of the left-handed neutrinos compared to the prediction that the right-handed neutrinos are very heavy, as a consequence of being singlets under the SM gauge group. The mass generated by the Yukawa interaction in Eq. (1.23) for Dirac neutrinos should be the same for both the left- and right-handed fields[2]. Looking at the Majorana mass model one also finds that unusually small Yukawa couplings are necessary to produce the small left-handed neutrino masses.

The See-saw mechanism provides a natural explanation for the large mass difference between the two chiralities using the combined Dirac-Majorana mass model of Eq. (1.45), where m_T is set to zero and m_S is assumed to be much larger than m_D. For simplicity only one generation will be considered here, where M_ν takes the form

$$M_\nu = \begin{pmatrix} 0 & m_D \\ m_D & m_S \end{pmatrix} . \tag{1.55}$$

As this is a 2×2 matrix it can be diagonalised approximately by the orthogonal matrix

$$U^\nu = \begin{pmatrix} 1 & m_D/m_S \\ -m_D/m_S & 1 \end{pmatrix} , \tag{1.56}$$

giving eigenvalues for the mass matrix M_ν of m_S and $-m_D^2/m_S$.[3] Instead of diagonalising the mass matrix M_ν, the left-handed column vector χ_L and right-handed column vector χ_R can be thought of as rotations of the projections of the fields η from Eq. (1.49)

$$\chi_L = U^\nu \eta_L \; ; \qquad \chi_R = U^\nu \eta_R \; . \tag{1.57}$$

which leads to

$$\eta_L \equiv \begin{pmatrix} \eta_{1L} \\ \eta_{2L} \end{pmatrix} = \begin{pmatrix} 1 & -m_D/m_S \\ m_D/m_S & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix}}_{\chi_L} \tag{1.58}$$

$$\eta_R \equiv \begin{pmatrix} \eta_{1R} \\ \eta_{2R} \end{pmatrix} = \begin{pmatrix} 1 & -m_D/m_S \\ m_D/m_S & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \nu_L^c \\ \nu_R \end{pmatrix}}_{\chi_R} , \tag{1.59}$$

and it can be seen that

$$\nu_L = \eta_{1L} + \frac{m_D}{m_S}\eta_{2L} \sim \eta_{1L} \; , \tag{1.60}$$

[2]The situation is very much different for Dirac neutrinos in models of extra dimensions. A new range of possibilities are described in [16].

[3]The minus sign can be removed by a chiral transformation of the respective neutrino field.

$$\nu_R = \eta_{2R} - \frac{m_D}{m_S}\eta_{1R} \sim \eta_{2R} \ . \tag{1.61}$$

The left-handed field ν_L is more or less the left-handed projection of a Majorana physical field with an associated light mass eigenvalue of $-m_D^2/m_S$ and the right-handed field ν_R is more or less the right-handed projection of a Majorana physical field with an associated heavy mass eigenvalue of m_S. Note that there is only one independent chiral projection of a Majorana spinor the other is found by charge conjugation and so from Eq. (1.60) it is found that there is minimal mixing of the left-handed field ν_L with the left-handed projection of the heavy Majorana physical field η_2.

When generalised to the three generation case $n_L,\ n_R = 3$ the diagonalised mass matrix $(M_\nu)_{\rm diag}$ takes the form

$$(M_\nu)_{\rm diag} = \left(\begin{array}{cc} (M_\nu)_{\rm light} & 0 \\ 0 & (M_\nu)_{\rm heavy} \end{array} \right) \ , \tag{1.62}$$

where $(M_\nu)_{\rm light}$ and $(M_\nu)_{\rm heavy}$ are 3×3 matrices defined by

$$(M_\nu)_{\rm light} = m_D^T m_S^{-1} m_D \ , \tag{1.63}$$

$$(M_\nu)_{\rm heavy} = m_S \ , \tag{1.64}$$

with m_D and m_S as 3×3 matrices also. The left-handed neutrino fields $\nu_{kL},\ k = \{1, 2, 3\}$ from Eq. (1.60) will become superpositions of the left-handed projections of the Majorana physical fields $\eta_i,\ i = \{1, ..., 6\}$

$$\nu_{kL} = \sum_i U^\nu_{ki} \eta_{iL} \ , \tag{1.65}$$

with the matrix U^ν_{ki} as a 3×6 matrix. However as mentioned previously the mixing of the left-handed fields ν_{kL} with the left-handed projections of three of the physical fields η_i from the heavy sector is minimal. Therefore U^ν_{ki} can be approximated as a 3×3 matrix[4]. Similarly the right-handed neutrino fields ν_{kR} will become superpositions of the right-handed projections of the Majorana physical fields η_i.

Again diagonalisation with the charged lepton unitary matrix U^ℓ is required and one obtains

$$\nu_{\ell L} = \sum_{i=1}^{3} \underbrace{(U^\ell)^\dagger_{\ell k} U^\nu_{ki}}_{U_{\rm mix}} \eta_{iL} \ . \tag{1.66}$$

Thus the spectrum of neutrino mass has been split up into a light sector of left-handed neutrinos and a heavy sector of right-handed neutrinos.

If it is assumed that $m_S \gg m_D$, which seems reasonable as m_S is not constrained by the scale of $SU(2)_L \times U(1)_Y$ breaking (~ 180 GeV) where on the

[4]This approximation will be reconsidered in Chapter 5.

other hand m_D is, then there exists a natural hierarchy to the mass spectrum of the neutrinos if m_S is around the Grand Unified scale $\sim 10^{16}$ GeV. Even more interesting though is that it can also give a natural hierarchy to the mass spectrum of all the leptons if one assumes that $m_D \sim m_\ell$ where m_ℓ is the charged lepton mass

$$(m_\nu)_{\text{light}} \sim \left(\frac{m_\ell^2}{m_S}\right) \qquad \ll m_\ell \ll \qquad (m_\nu)_{\text{heavy}} \sim m_S \ . \tag{1.67}$$

Here the assumption that m_T is zero can be justified by not allowing any extension of the current Higgs sector. As it happens the See-saw mechanism still works even if m_T is non-zero, but its value must be small. The assumption that m_D is of the scale of the charged lepton mass seems reasonable also, the Dirac matrix for all the other fermions is related to the electroweak scale (∼100 GeV) and the neutrinos should not be any different.

Chapter 2

Neutrino Oscillations

2.1 Leptonic mixing matrix

In 1958 it was first suggested by Pontecorvo [5] that neutrinos will oscillate in a vacuum if they are massive and mixed. The idea was subsequently developed as a two neutrino mixing model by Maki, Nakagawa and Sakata in 1962 [18]. The leptonic mixing matrix $U_{\rm mix}$ is therefore also known as the Maki-Nakagawa-Sakata matrix ($U_{\rm MNS}$). Maki, Nakagawa and Sakata showed that if it was assumed that the mass differences of the physical neutrinos were small, then the state of a neutrino produced in a weak interaction $|\nu_\alpha\rangle$ with momentum $p \gg m$ could be described as a coherent superposition of the mass eigenstates $|\nu_i\rangle$ associated with the physical neutrinos via $U_{\rm MNS}$

$$|\nu_\alpha\rangle = \sum_{i=1}^{3} (U_{\rm MNS})^*_{\alpha i} |\nu_i\rangle \ . \tag{2.1}$$

In general any $N \times N$ unitary matrix will have N^2 real parameters. If all the elements of U are real, then it will become an orthogonal matrix where there are $\frac{1}{2}N(N-1)$ independent real parameters, which can be given as the angles of a Euler rotation. The other $\frac{1}{2}N(N+1)$ real parameters must therefore be phases, which makes the general unitary matrix U complex. It is now necessary to make a distinction between $U_{\rm MNS}$ for Dirac and Majorana neutrinos by investigating these phases, as they become important when discussing CP violation in the leptonic sector.

The overall phase of a Dirac field is physically irrelevant, so that a redefinition of the phase of the neutrino field can be incorporated into $U_{\rm MNS}$ as follows

$$\nu_{\alpha L} \rightarrow e^{i\theta_\alpha} \nu_{\alpha L} \ \text{ corresponds to } \ U_{\alpha i} \rightarrow e^{-i\theta_\alpha} U_{\alpha i} \ . \tag{2.2}$$

Therefore N phases, each corresponding to one phase for each row can be absorbed from $U_{\rm MNS}$ into the neutrino fields $\nu_{\alpha L}$. This method can also be used on the charged lepton fields, where one phase for each column can be absorbed. However the charged leptons and neutrinos cannot be redefined with the same phase, as U_{MNS} will be unaffected overall. Therefore the $2N$ phases that have been absorbed must all be different. The total number of physical phases in U_{MNS} becomes

$$\frac{1}{2}N(N+1) - (2N-1) = \frac{1}{2}(N-1)(N-2) \;, \tag{2.3}$$

where these phases are responsible for CP violation in a similar way to that which occurs in the quark sector with the unitary matrix V_{CKM} [19]. It is obvious that $N \geqslant 3$ for CP violation to occur.

Majorana fields on the other hand are self-conjugate fields and therefore the phase change of the neutrino field must be opposite to that of its self-conjugate field. This means that the phase of a Majorana field cannot be redefined. Hence only the phase redefinitions from the charged leptons are allowed and the total number of physical phases in U_{MNS} becomes

$$\frac{1}{2}N(N+1) - N = \frac{1}{2}N(N-1) \;. \tag{2.4}$$

Now CP violation can occur if $N \geqslant 2$. The overall parameterisation for U_{MNS} where $N = 3$ can be achieved by a product of four matrices that include three Euler angles $\theta_{ij},\ 3 \geq j > i = \{1,2\}$, one Dirac phase δ and two Majorana phases β_1 and β_2

$$U_{\mathrm{MNS}} = R_{23} U_{13} R_{12} P_{12} \;, \tag{2.5}$$

where

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} , \qquad R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$
$$U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} , \qquad P_{12} = \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.6}$$

For convenience the notations $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ are used. These four matrices give U_{MNS} for Majorana neutrinos explicitly as

$$U_{\mathrm{MNS}} = \begin{pmatrix} c_{12}c_{13}e^{i\beta_1} & s_{12}c_{13}e^{i\beta_2} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}e^{i\beta_1} - c_{12}s_{23}s_{13}e^{i(\delta+\beta_1)} & c_{12}c_{23}e^{i\beta_2} - s_{12}s_{23}s_{13}e^{i(\delta+\beta_2)} & s_{23}c_{13} \\ s_{12}s_{23}e^{i\beta_1} - c_{12}c_{23}s_{13}e^{i(\delta+\beta_1)} & -c_{12}s_{23}e^{i\beta_2} - s_{12}c_{23}s_{13}e^{i(\delta+\beta_2)} & c_{23}c_{13} \end{pmatrix} \tag{2.7}$$

and for Dirac neutrinos one can set $\beta_1,\ \beta_2 = 0$ in the above matrix. Neglecting complex phases, the Euler rotations of U_{MNS} can be seen in Fig. (2.1). There are many other different parameterisations of U_{MNS} in the literature, for a more detailed look at this one should consult [20].

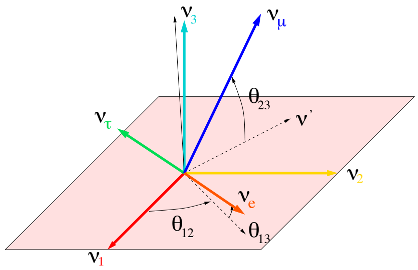

Figure 2.1: [21] A graphical illustration of the relationship between the weak interaction eigenstates and the mass eigenstates. Here the phases have been ignored so that the MNS matrix can be constructed as a product of three Euler rotations $U_{MNS} = R_{23}R_{13}R_{12}$, where $R_{13} = U_{13}$ without the phase angle δ. The values chosen for the angles are $\theta_{23} \approx \pi/4$, $\theta_{13} \lesssim 0.2$ and $\theta_{12} \approx \pi/6$. The reasons for these values will be made clear in the parameter values section.

2.2 Vacuum oscillations

As stated previously, the states of neutrinos created in a charged current interactions are the weak interaction eigenstates $|\nu_\alpha\rangle$. These are not the physical states, they are superpositions of the mass eigenstates $|\nu_i\rangle$ with associated masses m_i, where the mixing is described by Eq. (2.1)

$$|\nu_\alpha\rangle = \sum_i U^*_{\alpha i}|\nu_i\rangle \ , \tag{2.8}$$

with U as the MNS matrix. Although the 3-momentum $\mathbf{p}$ of the weak interaction eigenstates $|\nu_\alpha\rangle$ is fixed by momentum conservation of the interaction, the energies of the mass eigenstates $|\nu_i\rangle$ do not all have to be the same and in general they can be given by the relativistic energy-momentum relation

$$E_i = \sqrt{\mathbf{p}^2 + m_i^2} \ . \tag{2.9}$$

Dirac's equation for a neutrino wavefunction ν is written as

$$(i\gamma^\mu \partial_\mu - m)\nu = 0 \ . \tag{2.10}$$

Taking into account that during the evolution of a flavour state any particle-antiparticle mixing is negligible, spin projection stays the same and the neutrino is a relativistic particle, then a simpler evolution equation similar to the schrödinger equation can be used. After some time t, the evolution of the weak interaction eigenstate can be given by

$$|\nu_\alpha(t)\rangle = \sum_i e^{-iE_it} U^*_{\alpha i}|\nu_i\rangle \ , \tag{2.11}$$

where it is assumed the neutrinos ν_i are also stable particles. Therefore if the mass terms m_i are not all the same, then the weak interaction eigenstates will evolve differently as they will represent different superpositions of the mass eigenstates. As a result, a weak interaction eigenstate produced originally can develop into other weak interaction eigenstates. This is how neutrino oscillations are produced. To expand on this, one can write the amplitude of finding the state $|\nu_\beta\rangle$ from an original state $|\nu_\alpha\rangle$ as

$$\begin{aligned} \langle\nu_\beta|\nu_\alpha(t)\rangle &= \sum_{i,j}\langle\nu_j|U^T_{j\beta}e^{-iE_it}U^*_{\alpha i}|\nu_i\rangle \\ &= \sum_i e^{-iE_it}U_{\beta i}U^*_{\alpha i} \ , \end{aligned} \tag{2.12}$$

the last line uses the orthogonality of the mass eigenstates

$$\langle\nu_j|\nu_i\rangle = \delta_{ij} \ . \tag{2.13}$$

The probability for a transition to occur is then

$$\begin{aligned} P_{\nu_\alpha\nu_\beta}(t) = |\langle\nu_\beta|\nu_\alpha(t)\rangle|^2 &= \sum_{ij} e^{i(E_i-E_j)t}U^*_{\beta i}U_{\beta j}U_{\alpha i}U^*_{\alpha j} \\ &= \sum_i U^*_{\beta i}U_{\beta i}U_{\alpha i}U^*_{\alpha i} + \sum_{i\neq j} e^{i(E_i-E_j)t}U^*_{\beta i}U_{\beta j}U_{\alpha i}U^*_{\alpha j} \\ &= \sum_i U^*_{\beta i}U_{\beta i}U_{\alpha i}U^*_{\alpha i} + \sum_{i>j} U^*_{\beta i}U_{\beta j}U_{\alpha i}U^*_{\alpha j} 2\cos((E_i-E_j)t) \\ &= \delta_{\alpha\beta} - 4\sum_{i>j} U^*_{\beta i}U_{\beta j}U_{\alpha i}U^*_{\alpha j}\sin^2((E_i-E_j)t/2) \ . \end{aligned} \tag{2.14}$$

The masses of the neutrinos are assumed to be small compared to the momentum and therefore Eq. (2.9) can be expanded binomially as

$$E_i \simeq |\mathbf{p}| + \frac{m_i^2}{2|\mathbf{p}|} \ , \qquad |\mathbf{p}| \sim E \ . \tag{2.15}$$

The time t can be replaced by the distance (or baseline) L travelled by the neutrino and the probability for a transition becomes

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i>j} U^*_{\beta i}U_{\beta j}U_{\alpha i}U^*_{\alpha j}\sin^2\frac{\Delta m^2_{ij}L}{4E} \tag{2.16}$$

$$\begin{aligned} &= \delta_{\alpha\beta} - 4\sum_{i>j}\mathcal{R}e\left\{U^*_{\beta i}U_{\beta j}U_{\alpha i}U^*_{\alpha j}\right\}\sin^2\frac{\Delta m^2_{ij}L}{4E} \\ &\quad +2\sum_{i>j}\mathcal{I}m\left\{U^*_{\beta i}U_{\beta j}U_{\alpha i}U^*_{\alpha j}\right\}\sin\frac{\Delta m^2_{ij}L}{2E} \ . \end{aligned} \tag{2.17}$$

where Δm^2_{ij} is the mass difference squared

$$\Delta m^2_{ij} = m_i^2 - m_j^2 \ . \tag{2.18}$$

Here $\hbar = c = 1$, when they are put back in one finds

$$\frac{\Delta m^2_{ij}L}{4E} \simeq 1.27\frac{\Delta m^2_{ij}(\mathrm{eV}^2)L(m)}{E(\mathrm{MeV})} \ . \tag{2.19}$$

An expression for the probability of the antineutrino oscillation $P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)$ can be found by replacing U^* with U in Eq. (2.8) and following the same steps as above. It is evident from Eq. (2.17), that if there are no phase angles in $U_{\rm MNS}$ there will be no CP violation.

The explicit formulas for $P(\nu_\alpha \to \nu_\beta)$ in terms of the parameters of the $U_{\rm MNS}$ matrix are lengthy and not particularly needed, as they can in most cases of interest be approximated by oscillations among two weak interaction eigenstates. It is seen from oscillation experiments, that in the different scenarios which occur, each is dominated by a Δm^2 of different magnitude due to a series of approximations in the expressions for the probabilities. Therefore the mass eigenstates ν_i and ν_j corresponding to the neglected Δm^2_{ij} values will not mix appreciably and decoupling from the three state solution is acceptable to within certain limits of the theory [22]. Here the matrix U becomes a 2×2 mixing matrix and hence is orthogonal. Taking $|\nu_\alpha\rangle$ and $|\nu_\beta\rangle$ as the weak interaction eigenstates (eg. $\alpha = e,\ \beta = \mu,\ \tau$) one has

$$\begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} . \tag{2.20}$$

The probability term from Eq. (2.16) is still valid and becomes

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - \sin^2(2\theta) \sin^2 \frac{\Delta m^2 L}{4E} . \tag{2.21}$$

Here no phases were introduced, θ is the relevant mixing angle between the two weak interaction eigenstates and there is only one mass difference squared $\Delta m^2_{ij} = \Delta m^2_{12}$.

2.3 Matter oscillations

If neutrinos travel through matter, then the oscillation probabilities will change as a result of neutral and charged current scattering. This effect was first discussed by Wolfenstein [23] and later by Mikheyev and Smirnov in relation to solar neutrinos [24] and is therefore known as the MSW effect. This section follows closely with notes on Neutrino Mass and Mixing by Gonzalez-Garcia and Nir (2003) [25] in deriving the new mixing matrix in matter.

Only electrons, protons and neutrons are present in matter such as the earth or the sun, there are no muon or tau particles. While all weak interaction neutrinos ν_α will have the same interactions in matter due to the neutral current, giving no overall observable effect as a result, the ν_e states will experience a slightly different effective matter potential due to charged current interactions with electrons. The relevant matter effects for the ν_e states are produced from coherent effects due to forward elastic scattering. Here it is assumed that the momenta of the particles involved in the interactions are unchanged and any incoherent effects are found to be negligible.

The effective low energy Hamiltonian for the elastic scattering through the charged current interactions of electrons e and electron-neutrinos ν_e in a medium is given by

$$\begin{aligned}\mathcal{H}^{eff}_{CC} &= \frac{G_F}{\sqrt{2}}\{\overline{\nu_e}(x)\gamma_\mu(1-\gamma_5)e(x)\}\{\overline{e}(x)\gamma^\mu(1-\gamma_5)\nu_e(x)\} \\ &= \frac{G_F}{\sqrt{2}}\int d^3p_e f(E_e,T) \\ &\quad \times\Big\langle \langle e(s,p_e)|\overline{e}(x)\gamma^\mu(1-\gamma_5)\nu_e(x)\overline{\nu_e}(x)\gamma_\mu(1-\gamma_5)e(x)|e(s,p_e)\rangle\Big\rangle\end{aligned} \tag{2.22}$$

where $\Big\langle ... \Big\rangle$ denotes the averaging over electron spinors $\mathbf{s}$ and the summing over all electrons in the medium and $f(E_e,T)$ is the energy distribution function of electrons in the medium. Upon applying a Fierz transformation to the last line of Eq. (2.22) one obtains

$$\begin{aligned}\mathcal{H}^{eff}_{CC} &= \frac{G_F}{\sqrt{2}}\overline{\nu_e}(x)\gamma_\mu(1-\gamma_5)\nu_e(x) \\ &\quad \times\int d^3p_e f(E_e,T)\Big\langle \langle e(s,p_e)|\overline{e}(x)\gamma_\mu(1-\gamma_5)e(x)|e(s,p_e)\rangle\Big\rangle\end{aligned} \tag{2.23}$$

where s and p_e for the initial and final electron states are the same. The electron fields $e(x)$ in Eq. (2.23) can be expanded so that

$$\langle e(s,p_e)|\overline{e}(x)\gamma_\mu(1-\gamma_5)e(x)|e(s,p_e)\rangle =$$

$$\frac{1}{V}\langle e(s,p_e)|\overline{u_s}(p_e)a^\dagger_s(p_e)\gamma_\mu(1-\gamma_5)a_s(p_e)u_s(p_e)|e(s,p_e)\rangle\ , \tag{2.24}$$

where V has been introduced to normalize the volume. If it is assumed that the medium has an equal number of spin +1/2 and spin -1/2 electrons and using $a^\dagger_s(p_e)a_s(p_e) = \mathcal{N}^{(s)}_e(p_e)$ as the number operator, then applying the averaging/summing brackets $\Big\langle ... \Big\rangle$ gives

$$\frac{1}{V}\Big\langle \langle e(s,p_e)|a^\dagger_s(p_e)a_s(p_e)|e(s,p_e)\rangle\Big\rangle = N_e(p_e)\frac{1}{2}\sum_s\ , \tag{2.25}$$

with $N_e(p_e)$ as the number density of electrons with momentum p_e. One then obtains

$$\begin{aligned}\Big\langle \langle e(s,p_e)|\overline{e}(x)\gamma_\mu(1-\gamma_5)e(x)|e(s,p_e)\rangle\Big\rangle &= N_e(p_e)\frac{1}{2}\sum_s\overline{u_{(s)}}(p_e)\gamma_\mu(1-\gamma_5)u_{(s)}(p_e) \\ &= \frac{N_e(p_e)}{2}\mathrm{Tr}\Big[\frac{m_e+\not{p}}{2E_e}\gamma_\mu(1-\gamma_5)\Big] \\ &= N_e(p_e)\frac{p^\mu_e}{E_e}\ .\end{aligned} \tag{2.26}$$

It is also assumed here that the energy distribution function of the electrons $f(E_e, T)$ is homogeneous, isotropic and also normalised $\int d^3p_e f(E_e, T) = 1$. Isotropy of electrons in the medium implies $\int d^3p_e \vec{p_e} f(E_e, T) = 0$, hence only the p^0 term contributes to the integration when Eq. (2.26) is substituted into Eq. (2.23). With $\int d^3p_e f(E_e, T) N_e(p_e) = N_e$, Eq. (2.23) becomes

$$\mathcal{H}_{CC}^{eff} = \frac{G_F N_e}{\sqrt{2}} \overline{\nu_e}(x)\gamma_0(1-\gamma_5)\nu_e(x) \ . \tag{2.27}$$

The effective potential for the electron-neutrino as a result of charged current interactions is then given by

$$V_{CC}^{eff} = \langle \nu_e | \int d^3x \ \mathcal{H}_{CC}^{eff} | \nu_e \rangle = \frac{G_F N_e}{\sqrt{2}} \frac{2}{V} \int d^3x \ u_\nu^\dagger u_\nu = \sqrt{2} G_F N_e \ . \tag{2.28}$$

In the case of anti-neutrinos one obtains the same magnitude for the effective potential, but with the opposite sign in front.

As a result of this effective potential, the Hamiltonian in the weak interaction basis is modified. Again it is easier to analyse this in the two state case. The Hamiltonian H for the mass eigenstates ν_i is given as

$$H = \begin{pmatrix} E_1 & \\ & E_2 \end{pmatrix} \simeq |p| + \begin{pmatrix} \frac{m_1^2}{2|p|} & \\ & \frac{m_2^2}{2|p|} \end{pmatrix} , \tag{2.29}$$

where the evolution equation reads

$$i\frac{d}{dt}|\nu_i\rangle = H|\nu_i\rangle \ . \tag{2.30}$$

The Hamiltonian H' for the weak interaction eigenstates ν_α is then

$$\begin{aligned} H' &= UHU^\dagger \\ &= |p| + \frac{m_1^2 + m_2^2}{4|p|} + \frac{\Delta m^2}{4|p|} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} , \end{aligned} \tag{2.31}$$

and the angle θ for the mixing matrix U can be found as

$$\tan 2\theta = \frac{2H'_{12}}{H'_{22} - H'_{11}} \ . \tag{2.32}$$

Now if the effective potential from the charged current interactions is introduced only for the electron-neutrinos, one has from Eq. (2.31)

$$H'_M = |p| + \frac{m_1^2 + m_2^2}{4|p|} + \begin{pmatrix} -\frac{\Delta m^2}{4|p|}\cos 2\theta + V_{CC}^{eff} & \frac{\Delta m^2}{4|p|}\sin 2\theta \\ \frac{\Delta m^2}{4|p|}\sin 2\theta & \frac{\Delta m^2}{4|p|}\cos 2\theta \end{pmatrix} \tag{2.33}$$

The mixing matrix U used to diagonalise the original Hamiltonian H' to find the eigenvalues of the mass eigenstates does not work for H'_M. A new mixing matrix U_M is required

$$U_M = \begin{pmatrix} \cos\theta_M & \sin\theta_M \\ -\sin\theta_M & \cos\theta_M \end{pmatrix} \ . \tag{2.34}$$

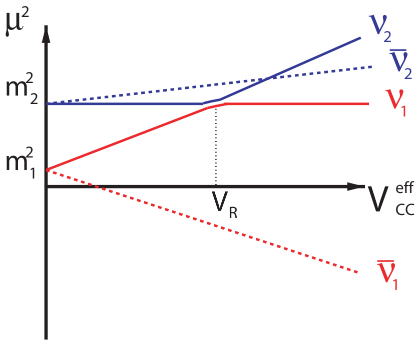

Figure 2.2: [25] A graphical illustration of level-crossing phenomenon. It was realised by Mikheyev and Smirnov that this was significant in understanding the neutrino flux from the sun.

The mass eigenstates ν_i are therefore rotations of the mass eigenstates in matter ν_i^M. The probability of mixing as given in Eq. (2.16) for vacuum oscillations can still be used but the angle θ is replaced with θ_M, where

$$\tan 2\theta_M = \frac{2(H'_M)_{12}}{(H'_M)_{22} - (H'_M)_{11}} = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - 2|p|V_{CC}^{eff}} \; . \tag{2.35}$$

The eigenvalues of H_M are

$$E_i = |p| + \frac{\mu_i}{2|p|} \; , \tag{2.36}$$

where the effective masses μ_i are given by

$$\mu_{1,2}^2(x) = \frac{m_1^2 + m_2^2}{2} + 2|p|V_{CC}^{eff} \mp \frac{1}{2}\sqrt{\left(\Delta m^2 \cos 2\theta - 2|p|V_{CC}^{eff}\right)^2 + \left(\Delta m^2 \sin 2\theta\right)^2} \; . \tag{2.37}$$

There is a resonance condition associated with the mixing angle θ_M, where it becomes maximal

$$\cos(2\theta) = \frac{2|p|}{\Delta m^2} V_{CC}^{eff} \; . \tag{2.38}$$

This condition produces an interesting effect known as level crossing shown in Fig. (2.2). Here $|\nu_e\rangle$ and $|\nu_\mu\rangle$ can be taken as the two weak interaction eigenstates. Then if the angle θ is very small ($\theta_M \simeq \frac{\pi}{4}$) and $V_{CC}^{eff} = 0$ corresponding to a vacuum one finds that the light eigenstate $|\nu_1\rangle$ is almost totally $|\nu_e\rangle$ and the heavier eigenstate $|\nu_2\rangle$ is almost totally $|\nu_\mu\rangle$. However when $V_{CC}^{eff} \gg \frac{\Delta m^2}{2|p|} \cos(2\theta)$ and θ becomes larger ($\theta_M \to \frac{\pi}{2}$) the light eigenstate $|\nu_1\rangle$ is almost totally $|\nu_\mu\rangle$ and the heavier eigenstate $|\nu_2\rangle$ is almost totally $|\nu_e\rangle$. This means that the effective mass of $|\nu_e\rangle$ starts low and then takes over $|\nu_\mu\rangle$ at a crossing point V_R and vice versa for $|\nu_\mu\rangle$.

In the three neutrino solution one can find the new mixing matrix U_M by solving

$$\underbrace{UHU^\dagger}_{H'} + V_{CC}^{eff} = \underbrace{U_M H_M U_M^\dagger}_{H'_M} \; . \tag{2.39}$$

which by neglecting $|p|$ in a three state extension of Eq. (2.29)[1] can be given by

$$U \begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix} U^\dagger + \begin{pmatrix} 2|p|V_{CC}^{eff} & & \\ & 0 & \\ & & 0 \end{pmatrix} = U_M \begin{pmatrix} \mu_1^2 & & \\ & \mu_2^2 & \\ & & \mu_3^2 \end{pmatrix} U_M^\dagger \, . \tag{2.40}$$

[1] As $|p|$ will contribute the same to all eigenstates.

Chapter 3

Parameter Values

3.1 Parameter space

There is very strong evidence now for neutrino oscillations from experiments and therefore that neutrinos do in fact have mass. The probability of oscillation for a two state solution was given previously as

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - \sin^2(2\theta)\sin^2\frac{\Delta m^2 L}{4E}\ , \tag{3.1}$$

where it was also mentioned that experiments commonly use this as opposed to the three state solution. This naive view turns out to fit the data quite well when the two state solution is converted into a three state solution [26, 27]. In Eq. (3.1), the parameters Δm^2, $\sin^2 2\theta$, L and E define the oscillation probability. $\sin^2 2\theta$ can be thought as giving the size of the oscillation and Δm^2 can be thought of as giving the dependence of L/E. From observing the fraction of different neutrinos present at detectors and comparing this to the expected values, probabilities can be observed. One can also fix the baseline L as well as measuring specific energies that the neutrinos have. With the probability, baseline and energy now fixed the parameter Δm^2 becomes a function of $\sin^2 2\theta$. It is therefore convenient to plot oscillation data in a $\sin^2 2\theta$-Δm^2 parameter space. One can then probe different regions of this graph by adjusting the baseline, as well as measuring the different energies of the neutrinos.

The parameter space is covered with $\Delta m^2 \geqslant 0$ and $0 \leqslant \theta \leqslant \frac{\pi}{2}$, or alternatively if either sign for Δm^2 is allowed, then $0 \leqslant \theta \leqslant \frac{\pi}{4}$ must hold on the condition that when $\Delta m^2 \to -\Delta m^2$, the transformation $\theta \to \frac{\pi}{2} - \theta$ should be made. This redefinition of the parameters has the same effect as redefining the mass eigenstates $\nu_1 \leftrightarrow \nu_2$. Here it is obvious the probability stays the same and in actual fact the probability stays the same under each of the parameter redefinitions separately. Therefore the mass squared differences Δm^2 found from the probability measured in oscillation experiments could be positive or negative. For example the $\nu_1 \leftrightarrow \nu_2$ transitions give Δm^2_{12}, but could also give $\Delta m^2_{21} = -\Delta m^2_{12}$. There is a two-fold discrete ambiguity in the interpretation of the oscillation probability and this has an effect on the mass hierarchy problem which shall be discussed in more detail in section 3.4.

3.2 Direct mass measurements

While oscillation experiments can help us determine the mass differences Δm^2, they do not tell us where on an energy scale to place the masses. Direct mass measurements have been carried out using β-decay, double β-decay and neutrinoless double β-decay in an attempt to find the upper limits on neutrino mass. In β-decay experiments, for example the Mainz experiment [28] which involves the decay of ^{3}H

$$^3\mathrm{H} \to {}^3\mathrm{He} + e^- + \bar{\nu}_e \ , \tag{3.2}$$

an upper limit has been observed

$$m_{\nu_e} < 2.2\mathrm{eV} \qquad (95\% \text{ Confidence Level CL}) \ . \tag{3.3}$$

Similar experiments, but less accurate have been performed for the ν_μ and ν_τ eigenstates. From studies of the $\pi^- \to \mu^- \bar{\nu}_\mu$ decay at PSI [29] it was found that

$$m_{\nu_\mu} < 190\mathrm{KeV} \qquad (90\% \text{ CL}) \ , \tag{3.4}$$

and studies of the $\tau^- \to \nu_\tau + 5\pi$ decay at LEP [30] it was found that

$$m_{\nu_\tau} < 18.2\mathrm{MeV} \qquad (95\% \text{ CL}) \ , \tag{3.5}$$

In double β-decay experiments [31] lepton number is conserved

$$Z \to (Z+2) + 2e^- + 2\bar{\nu}_e \ , \tag{3.6}$$

where Z is the number of protons. However more interesting experiments have been carried out using neutrinoless double β-decay

$$Z \to (Z+2) + 2e^- \ . \tag{3.7}$$

This violates lepton number and would not be allowed if neutrinos are Dirac particles. This special type of decay can only occur when there is a $\bar{\nu}\nu$ annihilation allowed by the presence of a Majorana mass term. At the moment there are only limits on this type of double β-decay, but the strongest evidence comes from the Heidelberg-Moscow group [32], which suggests that the effective Majorana mass of the ν_e neutrino is

$$m_{ee} < 0.34\mathrm{eV} \qquad (90\% \text{ CL}) \ , \tag{3.8}$$

there are however large uncertainties associated with these experiments. The NEMO3 experiment currently running is expected to reach a sensitivity of $m_{ee} \sim$ 0.1 eV in the near future [33].

3.3 Oscillation experiments

3.3.1 Atmospheric neutrinos

In the upper atmosphere cosmic rays interact with nuclei to produce pions. These pions then decay into muons, which then decay to electrons and neutrinos are also produced

$$\begin{aligned}\pi^+ &\rightarrow \mu^+\nu_\mu \\ &\quad \mu^+ \rightarrow e^+\bar{\nu}_\mu\nu_e \ .\end{aligned} \tag{3.9}$$

There is a similar chain for the particle conjugate reaction involving π^-. It is clear that there will be twice as many muon neutrinos as electron neutrinos expected at detectors which measure the neutrino fluxes from their charged current interactions with protons and neutrons. The experiments carried out so far have all measured a deficit of ν_μ neutrinos arriving at detectors. This atmospheric anomaly has been observed by Kamiokande, IMB, Soudan, MACRO and Superkamiokande in Japan as the most famous one, with the highest statistics. Here the range of neutrino energies recorded using water Cerenkov detectors is from 5 MeV up to $\sim$ GeV, where they are detected by their charged current interaction with protons and neutrons

$$\begin{aligned}&\nu_e + n \rightarrow e^- + p \ ; \quad \nu_\mu + n \rightarrow \mu^- + p \\ &\bar{\nu}_e + p \rightarrow e^+ + n \ ; \quad \bar{\nu}_\mu + p \rightarrow \mu^+ + n \ .\end{aligned} \tag{3.10}$$

It is thought that a ν_μ state oscillates into a ν_β state, where it is most likely that $\beta = \tau$. Transitions to the ν_e state are excluded, as the fluxes for these at the detectors match that of the theoretical predictions. Also the null results of the Palo Verde [34] and the CHOOZ reactor [35] experiments suggest that the contribution from ν_e transitions will be negligible. However a transition to some sterile neutrino ν_s cannot be overlooked. As a result of these observations the two state solution for neutrino oscillations can be used in approximations and the Δm^2 value measured is commonly associated with the Δm^2_{23} value of the $\nu_2 \leftrightarrow \nu_3$ mass eigenstate transition.

$$\begin{aligned}(\nu_\mu \rightarrow \nu_\tau) \quad : \quad &\Delta m^2_{23} \sim 3 \times 10^{-3} \ \mathrm{eV}^2 \\ &\sin^2 2\theta_{23} \sim 1\end{aligned} \tag{3.11}$$

The parameter region probed in several experiments is shown in Fig. (3.1) where the allowed regions are shaded.

3.3.2 Solar neutrinos

There are many β-decays and inverse β-decays within the sun that produce ν_e and $\bar{\nu}_e$. However there is a large discrepancy between the data detected and

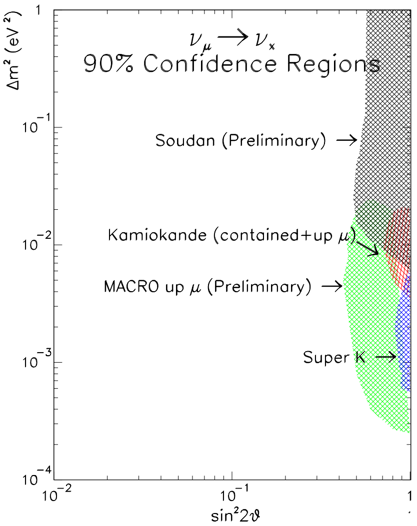

Figure 3.1: [10] A $\sin^2 2\theta$-Δm^2 graph showing the allowed regions for Atmospheric oscillations from several experiments.

that predicted from the Standard Solar Model (SSM) [36]. The SSM predicts the neutrino energy spectrum coming from the sun in terms of the flux of neutrinos from each type of reaction. In the case of water Cerenkov detectors these neutrinos must all have $E > 5$ MeV and therefore virtually all of them will come from ^8B decay within the sun

$$ {}^{8}_{5}\mathrm{B} \rightarrow {}^{8}_{4}\mathrm{Be} + e^+ + \nu_e \ , \tag{3.12} $$

giving an energy range for neutrinos of up to 20 MeV. Solar neutrino experiments such as Homestake, Kamiokande, Superkamiokande, SAGE, GALEX/GNO and SNO in Canada as the most famous, all measure approximately half the expected rate of ν_e's. The difference between SNO and Superkamiokande detectors is that SNO is a heavy water Cerenkov detector and so can also measure the neutral current interaction.

$$ \begin{aligned} \mathrm{CC} &: \quad \nu_e + d \rightarrow p + p + e^- \\ \mathrm{NC} &: \quad \nu_\beta + d \rightarrow p + n + \nu_\beta \ , \end{aligned} \tag{3.13} $$

where d is deuterium and $\beta = \{e,\ \mu,\ \tau\}$. It is thought that a ν_e state oscillates into a ν_β state, where $\beta = \mu,\ \tau$. There are two distinct oscillation solutions for solar neutrinos: The MSW solution which accounts for matter induced oscillations in the sun [37] and the VAC solution which accounts only for the vacuum oscillations outside the sun. ν_e's reaching the earth's upper atmosphere have shown no evidence of oscillation in the region between the upper atmosphere and detectors, it is therefore presumed that their oscillation length is much greater than 13,000 km.

The MSW solution has three main scenarios: Adiabatic [38] which has an associated Large Mixing Angle (LMA), non-adiabatic [39] which has an associated Small Mixing Angle (SMA) and finally the low mass (LOW) scenario. The Δm^2 value measured is commonly associated with the Δm^2_{12} value of the $\nu_1 \leftrightarrow \nu_2$ mass eigenstate transition. Quoting results from [40]

$$\begin{array}{lcll} \text{MSW } (\nu_e \to \nu_\beta) & : & \text{LMA} & \Delta m^2_{12} \sim 4.5 \times 10^{-5} \text{ eV}^2 \\ & & & \sin^2 2\theta_{12} \sim 0.82 \\ & & \text{SMA} & \Delta m^2_{12} \sim 4.7 \times 10^{-6} \text{ eV}^2 \\ & & & \sin^2 2\theta_{12} \sim 1.6 \times 10^{-3} \\ & & \text{LOW} & \Delta m^2_{12} \sim 1 \times 10^{-7} \text{ eV}^2 \\ & & & \sin^2 2\theta_{12} \sim 0.97 \end{array} \tag{3.14}$$

$$\begin{array}{lcl} \text{VAC } (\nu_e \to \nu_\beta) & : & \Delta m^2_{12} \sim 4.6 \times 10^{-10} \text{ eV}^2 \\ & & \sin^2 2\theta_{12} \sim 0.83 \end{array} \tag{3.15}$$

The most favoured solution at the moment is the LMA solution due to results from the SNO experiment [41] and more recently the KamLAND experiment [42]. Before KamLAND it was thought that there may be better solutions to the solar neutrino deficit problem from Non-Standard Neutrino Interactions (NSNI) and that they provided a better fit with the data than the LMA solution [43]. Also more interesting was that the atmospheric neutrino data could also be incorporated into the NSNI solution [44] and because NSNI worked for massless neutrinos it meant that these oscillation experiments may have not necessarily implied massive neutrinos after all. However it is now widely accepted that neutrino mixing due to mass is the leading mechanism for the atmospheric and solar neutrinos. NSNI may still be relevant in other oscillation experiments though, but at higher energies, where they may be comparable to the oscillation effects. NSNI will be discussed in more detail in the next chapter.

3.3.3 Accelerator neutrinos

The weakest hint of oscillations so far is that from accelerators. In particular the Liquid Scintillating Neutrino Detector (LSND) experiment which looked for $\bar{\nu}_\mu$ conversion into $\bar{\nu}_e$. The $\bar{\nu}_\mu$ neutrinos were produced using a 800 MeV proton beam incident on the LAMPF beam stop, which generated pions and neutrinos were produced from the subsequent decay chain similar to that in the earth's upper atmosphere.

$$\begin{array}{rcl} \pi^+ & \to & \mu^+ \nu_\mu \\ & & \mu^+ \to e^+ \bar{\nu}_\mu \nu_e \ . \end{array} \tag{3.16}$$

The $\bar{\nu}_\mu$'s produced had a maximum energy of $\sim$ 54 MeV and their flux was known as the Decay at Rest (DAR) flux as most of the π^+'s would come to rest

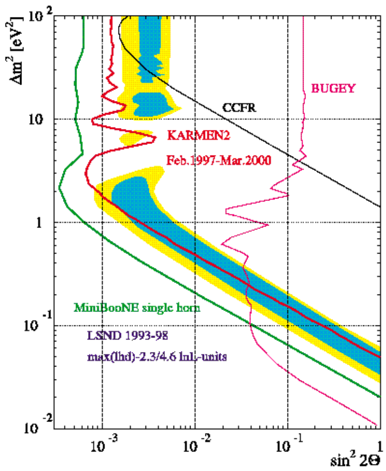

Figure 3.2: [10] A $\sin^2 2\theta$-Δm^2 graph showing the allowed regions for Accelerator oscillations from several experiments.

and decay through the sequence in Eq. (3.16). The DAR flux was therefore used to study $\bar{\nu}_\mu \to \bar{\nu}_e$ oscillations. However 3% of the π^+'s Decayed in Flight (DIF). This DIF flux was used to study $\nu_\mu \to \nu_e$ oscillations. For DAR measurements the $\bar{\nu}_e$'s were detected in the process

$$\bar{\nu}_e + p \to e^+ + n \ , \tag{3.17}$$

and for DIF measurements the ν_e's were observed via the detection of electrons produced in the process

$$\nu_e + C \to e^- + X \ . \tag{3.18}$$

The data suggested the appearance of $\bar{\nu}_e$ and the final outcome before the experiment ended in 1998 was

$$P(\bar{\nu}_\mu \to \bar{\nu}_e) = (2.6 \pm 0.8) \times 10^{-3} \ . \tag{3.19}$$

However the KARMEN [45] and BUGEY experiments rule out large regions of the $\sin^2 2\theta$-Δm^2 parameter space allowed by LSND.

The Δm^2 value measured is commonly associated with the Δm^2_{13} value of the $\nu_1 \leftrightarrow \nu_3$ mass eigenstate transition. The experiments so far suggest the remaining allowed region is

$$\begin{aligned} (\bar{\nu}_\mu \to \bar{\nu}_e) \quad : \quad & \Delta m^2_{13} \sim 0.2 \to 2 \text{ eV}^2 \\ & \sin^2 2\theta_{13} \sim 2 \times 10^{-3} \to 4 \times 10^{-2} \end{aligned} \tag{3.20}$$

which is shown graphically in Fig. (3.2). The problem with the LSND experiment is that it is the only one to have observed this anomaly of excess $\bar{\nu}_e$'s and subsequently it is left out of many theoretical models. An experiment called

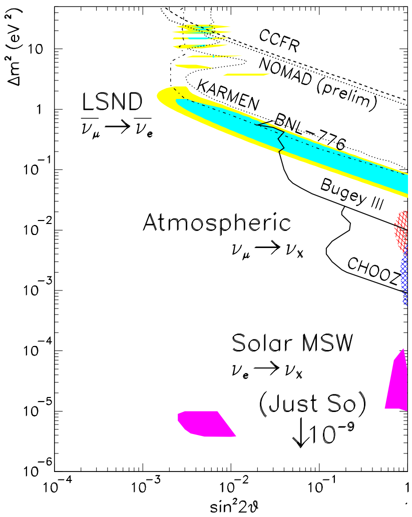

Figure 3.3: [10] A $\sin^2 2\theta$-Δm^2 graph showing the allowed regions for All oscillation experiments.

MiniBoone [46] has been designed to cover the region of parameter space indicated by LSND and give some conclusive evidence for or against this type of oscillation.

3.4 Mass hierarchy

The Atmospheric, Solar and Accelerator experiments are three pieces of evidence for neutrino oscillations that give three Δm^2 values all of different magnitude. However in a three neutrino state framework for oscillations there can only be two Δm^2 values of different magnitude, as they must obey

$$\Delta m_{21}^2 + \Delta m_{32}^2 + \Delta m_{13}^2 = 0. \tag{3.21}$$

There is no acceptable explanation at present that links the three types of experiments together within this three neutrino state framework. There are many experiments that agree with the Solar and Atmospheric neutrino oscillation theories, but the Accelerator neutrino oscillation data is more sceptical, as only the LSND experiment has observed this. If the data is confirmed by the MiniBoone experiment and perhaps eventually the Boone experiment, then it may be necessary to have a four neutrino state framework for oscillations to describe the data [49].

Assuming that $\Delta m_{32}^2 \simeq \Delta m_{13}^2$ for the moment, then there are two magnitudes involved: $\Delta m_{\odot}^2$ from the solar neutrino experiments and Δm_{atm}^2 from the atmospheric experiments which enable Eq. (3.21) to be solved. There are two

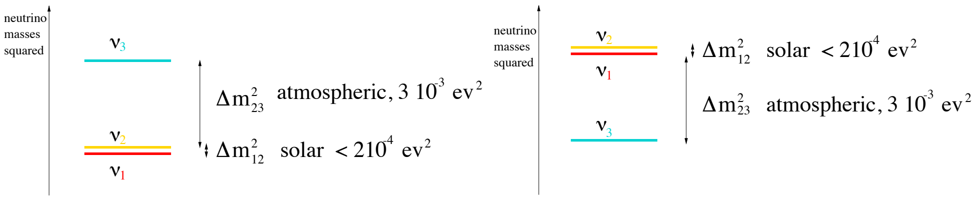

Figure 3.4: [21] A figure showing the two schemes available for solving the mass differences equation (Eq. (3.21)), Direct (left) and Inverted (right).

ways in which this can be achieved

$$\Delta m^2_{21}\ (= \Delta m^2_{\odot}) \quad \ll \quad \Delta m^2_{32} \simeq \Delta m^2_{31}\ (= \Delta m^2_{\text{atm}}) > 0 \quad \text{or} \tag{3.22}$$

$$\Delta m^2_{21}\ (= \Delta m^2_{\odot}) \quad \ll \quad -\Delta m^2_{31} \simeq -\Delta m^2_{32}\ (= |\Delta m^2_{\text{atm}}|) > 0 \tag{3.23}$$

Eq. (3.22) is referred to as the direct scheme and Eq. (3.23) is referred to as the inverted scheme, see Fig. (3.4). For the direct scheme there are two scenarios: the hierarchal mass direct scheme with $m_1 \ll m_2 \ll m_3$ where $m_2 \simeq \sqrt{\Delta m^2_{21}}$ and $m_3 \simeq \sqrt{\Delta m^2_{32}}$. And the quasi-degenerate mass direct scheme with $m_1 \simeq m_2 \simeq m_3 \gg \Delta m^2_{21}, \Delta m^2_{32}$. For the inverted scheme there is only one scenario and it implies $m_3 < m_1 \simeq m_2$.

3.5 Neutrino Factories

3.5.1 Basics of a neutrino factory

The current experiments that give us evidence for neutrino oscillations and also an idea of some of the mixing parameters of the leptonic mixing matrix are not enough to completely determine all of its parameters. There are several other questions of importance too, such as whether CP violation occurs in the leptonic sector, what the neutrino mass scheme is and if sterile neutrinos exist. It is therefore necessary to ask about the possibility of producing higher energy and higher intensity neutrino beams than those provided by conventional means using pion decay. The answer to this may be a "Neutrino Factory".

The concept behind a neutrino factory [50] lies in the development of intense bright μ^+ and μ^- beams. These would not only open the door for multi-TeV muon colliders, muon-proton colliders etc., but would also provide high energy neutrino beams, as all of the muons would decay to produce neutrinos.

Conventional neutrino beams are made from almost completely $\nu_\mu(\bar{\nu}_\mu)$, which are produced from the decay of charged pions $\pi^+ \to \mu^+\nu_\mu$ ($\pi^- \to \mu^-\bar{\nu}_\mu$). A small percentage of ν_e is also produced from three body kaon decays, but it is not enough to study $\nu_e \to \nu_X$ oscillation in any detail.

With a very intense muon source it is possible to obtain high energy ν_e ($\bar{\nu}_e$) and ν_μ ($\bar{\nu}_\mu$) beams from the decays $\mu^+ \to e^+\nu_e\bar{\nu}_\mu$ and $\mu^- \to e^-\bar{\nu}_e\nu_\mu$. Then

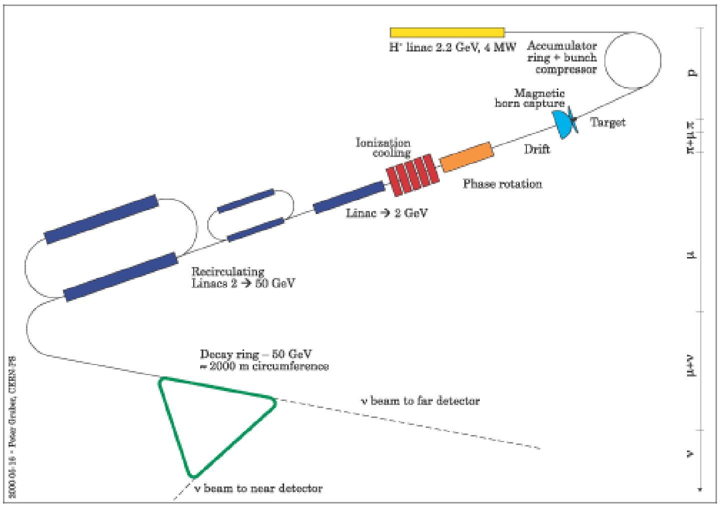

Figure 3.5: [52] Diagram of a possible neutrino factory at CERN

with only a millimole of muons per year, it is expected that high energy beams containing $\sim \mathcal{O}(10^{20})$ neutrinos and antineutrinos could be produced [50].

However muons live 100 times longer than charged pions. This and the fact that a large decay fraction f is required, means that either a linear decay channel of tens of kilometers long, or a storage ring is needed. It is therefore more practical to inject the muons after being rapidly accelerated, into a storage ring with long straight sections and it has been shown that with this method $f \sim 0.3$ is achievable [51].

This storage ring can then be tilted downwards so that the resulting neutrino beam can pass through the earth, allowing for a long baseline $L \sim \mathcal{O}(10^4)$ km. There are two types of neutrino factory under consideration at the moment:

- Entry level: Providing 20 → 30 GeV, with $\sim \mathcal{O}(10^{19})$ muon decays/year giving $\sim \mathcal{O}(10^{21})$ kt-decays after a few years with a 50 kt detector at 50% efficiency.

- High performance: Providing 50 GeV, with $\sim \mathcal{O}(10^{20})$ muon decays/year giving $\sim \mathcal{O}(10^{22})$ kt-decays after a few years with a 50 kt detector at 50% efficiency.

3.5.2 Benefits

There are numerous benefits from building a neutrino factory for neutrino physics [50, 53]. The main benefit, as mentioned previously is that it would provide ν_e ($\bar{\nu}_e$) and ν_μ ($\bar{\nu}_\mu$) beams at very high energies and event rates[1].

Other benefits include:

- Measurements $\sim \mathcal{O}(10^6)$ events per year per kg for a near detector at a high performance neutrino factory. This would then give rise to a new era in non-oscillation experiments.

- The energy spectrum of neutrinos from a neutrino factory would be much narrower than that from conventional beams. As the muon decay spectrum is very well known, the systematic uncertainties of the flux and spectrum of the neutrinos for long baselines could be expected to be much smaller than those of a conventional beam.

- The intensity of the beams would be sufficiently high, that baselines of the order of the earth's diameter could be used. A neutrino factory could then be built at CERN (see Fig. (3.5)) and the beam directed at a detector in the US or Japan. This would allow a detailed study of the MSW effect, as well as a complete determination of the neutrino mixing matrix, CP violation and whether any sterile neutrinos are present.

- It would be the first step toward a muon collider.

It seems that there are compelling reasons to want to build a neutrino factory and one could very well be built within the next ten years. It is therefore necessary to take a look at any possible sources of new physics that may be detectable using these higher energy neutrinos beams. This shall be the subject for the next two chapters.

[1] A factor of 60 larger than that of the next generation of conventional beams eg. NUMI at FNAL [50].

Chapter 4

Non-Standard Neutrino Interactions

4.1 Overview

Neutrino oscillation experiments are designed to investigate physics beyond the SM by giving an indication that neutrinos mix and therefore have mass as a result. However when the data from detectors is interpreted, it is assumed that neutrinos interact with other particles within the framework of the SM. If extensions of the SM are being investigated, then it seems reasonable to enquire into the possibility of interactions from New Physics (NP)[1], where the effect would be seen through Non-Standard Neutrino Interactions (NSNI). In his seminal paper Wolfenstein [23] suggested that NSNI in matter could also give the effect of neutrino oscillations occurring.

If NSNI do turn out to contribute significantly to the neutrino interactions, then any interpretations of data taken in oscillation experiments will need to take them into account. Within this context it would be possible to have an electron-neutrino produce a muon in a detector via some NP and thus it may be concluded that an oscillation has occurred, when it actual fact no oscillation has taken place. It is therefore necessary to break down neutrino experiments into three stages:

- The production process
- The time evolution
- The detection process

[1]The Minimal Supersymmetric Standard Model (MSSM) without R-parity is one such extension that could provide NP.

The NSNI can then be incorporated into each part separately, allowing a thorough analysis to be undertaken by understanding their effect at the different stages. To start this chapter it is necessary to look within the context of a particular experiment, namely that of a neutrino factory experiment. However the same approach can be used for experiments with a conventional beam. This chapter follows closely to material in a paper by Ota et al. (2001) [54].

4.2 Formalism

4.2.1 Observations at a neutrino factory

At a neutrino factory experiment muons with negative charge would decay at an accumulate ring and wrong sign muons would be observed in a detector at a baseline L km away. A sketch of this can be seen in Fig. (4.1). A wrong sign

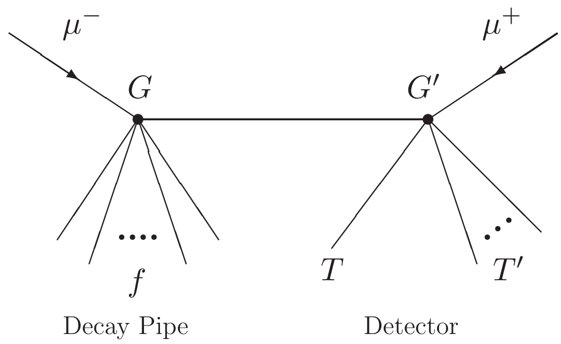

Figure 4.1: [54] What would really be seen in a neutrino factory.

muon at the detector would normally be interpreted as evidence for $\bar{\nu}_e \to \bar{\nu}_\mu$ as shown in Fig. (4.2) below, where the $\bar{\nu}_e$ is produced in the weak interaction $\mu^- \to e^- \nu_\mu \bar{\nu}_e$. However if NSNI are permitted in the production process eg.

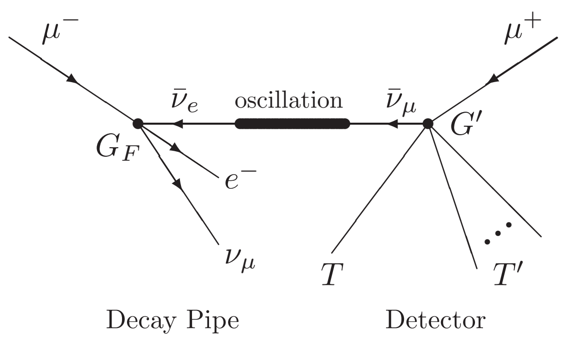

Figure 4.2: [54] Standard interpretation of a wrong sign event.

$\mu^- \to e^- \nu_\mu \bar{\nu}_\alpha$, as there may be other heavier carriers of the charged current besides the W boson, then effective four fermion operators of the form

$$\lambda_\alpha(\bar{e}\gamma_\mu\mu)(\bar{\nu}_\mu\gamma^\mu\nu_\alpha), \quad \alpha \neq e \ , \tag{4.1}$$

would be present. These would also give a wrong sign muon at the detector as shown in Fig. (4.3) and there would then be no way to distinguish between the two types of production processes.

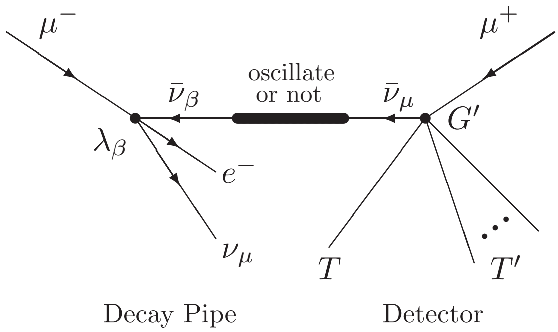

Figure 4.3: [54] Diagram which gives same signal as that given by Fig. (4.2).

To obtain a probability amplitude of the whole process including known physics and NP, all possible processes must be summed and then the square of the summation must be taken. This results in an interference phenomenon between the different interactions that can occur (weak and NSNI). Although this may seem like it will confuse matters, in actual fact the interference produces an enhancement of the effect of NP. Before investigating the three stages mentioned at the start of the chapter, it is necessary to think about a neutrino experiment as a whole system, once this is done it can then be broken down into the separate stages.

4.2.2 Quantum mechanical interference

To begin understanding this interference, a quantum mechanical treatment of a basic system can be analysed and then applied to a particular experimental process, in our case a neutrino factory experiment.

A system comprised of an initial state $\mathcal{A} + \mathcal{T}$ and final state $\mathcal{C} + \mathcal{U}$ via an intermediate state $\mathcal{B}$ has a transition probability given by $\Phi(\mathcal{A}, \mathcal{T}; \mathcal{B}; \mathcal{C}, \mathcal{U})$. The initial state $\mathcal{T}$, final state $\mathcal{U}$ and the intermediate state $\mathcal{B}$ are set as unobservables and the transition probability is given by

$$P(\mathcal{A} + \mathcal{T} \to \mathcal{C} + \mathcal{U}) = \left| \sum_{\mathcal{B}} \Phi(\mathcal{A}, \mathcal{T}; \mathcal{B}; \mathcal{C}, \mathcal{U}) \right|^2 . \tag{4.2}$$

To obtain the transition probability for $\mathcal{A} \to \mathcal{C}$, the unobserved initial states $\mathcal{T}$

and final states $\mathcal{U}$ must be summed up

$$P(\mathcal{A} \to \mathcal{C}) = \sum_{\mathcal{T},\mathcal{U}} P(\mathcal{A} + \mathcal{T} \to \mathcal{C} + \mathcal{U}). \tag{4.3}$$

Supposing now that the amplitude $\Phi(\mathcal{A}, \mathcal{T}_0; \mathcal{B}_0; \mathcal{C}, \mathcal{U}_0)$ is dominant over the other amplitudes with $\mathcal{B}(\neq \mathcal{B}_0)$, $\mathcal{T}(\neq \mathcal{T}_0)$ and $\mathcal{U}(\neq \mathcal{U}_0)$. The dominant part of the probability $P(\mathcal{A} \to \mathcal{C})$ would then become $P(\mathcal{A} + \mathcal{T}_0 \to \mathcal{C} + \mathcal{U}_0)$

$$\begin{aligned} P(\mathcal{A} \to \mathcal{C}) &\simeq P(\mathcal{A} + \mathcal{T}_0 \to \mathcal{C} + \mathcal{U}_0) \qquad (4.4) \\ &= |\Phi(\mathcal{A}, \mathcal{T}_0; \mathcal{B}_0; \mathcal{C}, \mathcal{U}_0)|^2 \\ &\quad + 2\mathrm{Re}\left[\Phi(\mathcal{A}, \mathcal{T}_0; \mathcal{B}_0; \mathcal{C}, \mathcal{U}_0)^* \sum_{\mathcal{B} \neq \mathcal{B}_0} \Phi(\mathcal{A}, \mathcal{T}_0; \mathcal{B}; \mathcal{C}, \mathcal{U}_0)\right] \\ &\quad + \left|\sum_{\mathcal{B} \neq \mathcal{B}_0} \Phi(\mathcal{A}, \mathcal{T}_0; \mathcal{B}; \mathcal{C}, \mathcal{U}_0)\right|^2 . \qquad (4.5) \end{aligned}$$

The leading amplitude of the first term in Eq. (4.5) will undoubtedly come from known physics, whereas the other amplitudes of the second and third terms will include contributions from NP.

The third term will contain only NP and will be almost negligible. However even if this is the case and the NP is not detectable, most probably due to strong experimental limits in a direct measurement, it may be possible to see an enhanced effect of the NP processes from the interference of the known physics and NP in the second term.

The second term gives the interference between the leading amplitude and the subleading amplitudes. It should be noted that this is a result of interference between processes with the same[2] initial and final states.

In general for a neutrino experiment, $\mathcal{A}$ would correspond to the state of the parent particle of the neutrinos produced, $\mathcal{T}$ would correspond to the state of the target particle that interacts with a neutrino in the detector and $\mathcal{C}$ would correspond to the state of a particle which produces an identifiable event in the detector as a result of the interaction with $\mathcal{T}$. The states $\mathcal{U}$ are those from all unobserved particles appearing at both the production and detection process and the intermediate state $\mathcal{B}$ is that of a neutrino particle.

As shown in Fig. (4.3) an intermediate neutrino state may or may not oscillate and so the first term in Eq. (4.5) is given by the physics of the weak interaction and any oscillations that might occur.

The different intermediate states $\mathcal{B}(\neq \mathcal{B}_0)$ induced by NSNI will result in the contribution of non-vanishing amplitudes to the total process in the form of

[2]Here "same state" signifies a state with the same particle species and corresponding physical quantities such as energy.

the third term and also to the interference in the second term. These new interactions will violate flavour conservation and also most probably the standard chiral property of the four fermion operators.

The interference of the leading amplitude of the weak interaction (and perhaps oscillations) with that of NSNI in the second term will result in sub-leading contributions to the total probability and may not be negligible. The result in either case of oscillation or not is the significant contribution of NP to the probability amplitude.

4.2.3 Interference at a neutrino factory

For a neutrino factory experiment the initial state $\mathcal{A}$ is μ^- and the final state $\mathcal{C}$ is μ^+. Fig. (4.4) shows a graphical representation of the process. The other states are given by $\mathcal{T} \leftrightarrow T$ and $\mathcal{U} \leftrightarrow D + T'$.

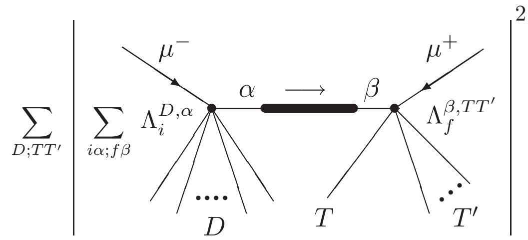

Figure 4.4: [54] Transition rate for "$\bar{\nu}_e \to \bar{\nu}_\mu$".

The NP at the source and detector can be parameterised by the four fermion effective couplings $(G^s_{\rm NP})_{\alpha\beta}$ and $(G^d_{\rm NP})_{\alpha\beta}$ respectively with $\alpha,\ \beta = \{e,\ \mu,\ \tau\}$. Here $(G^s_{\rm NP})_{\alpha\beta}$ refers to an interaction where a ν_β is produced in association with an incoming α^- or outgoing α^+ charged lepton and $(G^d_{\rm NP})_{\alpha\beta}$ refers to an interaction where an incoming ν_β produces an α^- charged lepton. It is also necessary to introduce the normal SM flavour conserving effective four fermion coupling $G_{\rm F}\delta_{\alpha\beta}$, where $\alpha = \beta$.

The effective four fermion operator at the production process $\mu^- \to e^-\nu_\mu\bar{\nu}_e$ for SM physics involves only the weak interaction, which has the lorentz structure $(V-A)(V-A)$

$$-2\sqrt{2}G_F(\bar{\nu}_\mu\gamma_\rho L\mu)(\bar{e}\gamma^\rho L\nu_e)\ . \tag{4.6}$$

For NSNI it is possible to have both a $(V-A)(V-A)$ and $(V-A)(V+A)$ effective four fermion operator for the production process $\mu^- \to e^-\nu_\mu\bar{\nu}_\alpha$

$$-2\sqrt{2}(G^s_{\rm NP})_{e\alpha}(\bar{\nu}_\mu\gamma_\rho L\mu)(\bar{e}\gamma^\rho L\nu_\alpha)\ , \qquad \alpha = e,\ \mu,\ \tau\ , \tag{4.7}$$

$$-2\sqrt{2}\lambda_\alpha(\bar{\nu}_\mu\gamma_\rho L\nu_\alpha)(\bar{e}\gamma^\rho R\mu)\ , \qquad \alpha = e,\ \mu,\ \tau\ . \tag{4.8}$$

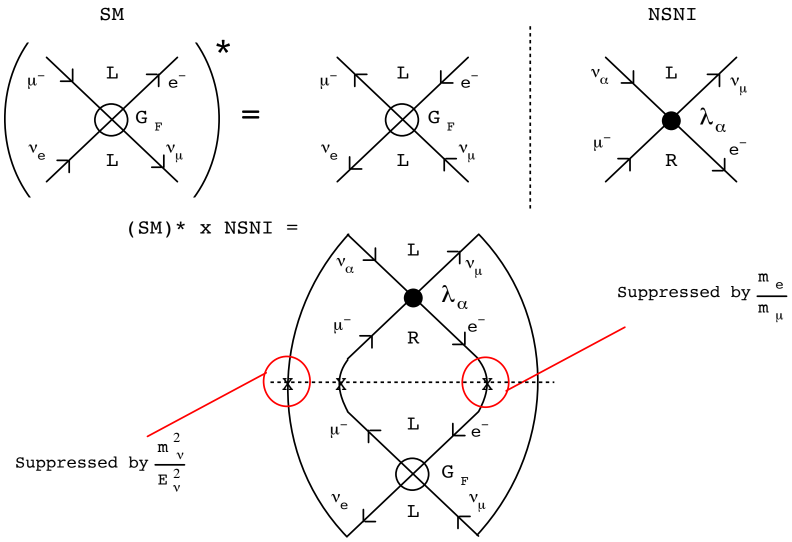

Figure 4.5: Suppression of subleading order term for $(V-A)(V+A)$ NSNI structure. The m_ν^2/E_ν^2 suppression for the neutrinos is a result of the m_ν/E_ν term only allowing a L/R interaction, this is only present if $\alpha \neq e$.

From Eq. (4.5) the term of interest is the interference of the SM and NSNI terms

$$P = |\Phi|^2 \simeq \underbrace{|\mathrm{SM}|^2}_{\text{leading order}} + \underbrace{2\mathcal{R}e\left\{(\mathrm{SM})^*(\sum \mathrm{NSNI})\right\}}_{\text{subleading order}} + \underbrace{(\sum \mathrm{NSNI})^2}_{\text{negligible}} \; . \tag{4.9}$$

However the interference consisting of

$$\underbrace{(V-A)(V-A)^*}_{\mathrm{SM}}\underbrace{(V-A)(V+A)}_{\mathrm{NSNI}} \tag{4.10}$$

is highly suppressed by m_e/m_μ due to a difference in the chirality dependence of the NSNI part to that of the SM, see Fig. (4.5). This introduces complications to treating NSNI of the $(V-A)(V+A)$ structure in any normal fashion. But as it is highly suppressed one can neglect their effect for most practical purposes. On the other hand the interference consisting of

$$\underbrace{(V-A)(V-A)^*}_{\mathrm{SM}}\underbrace{(V-A)(V-A)}_{\mathrm{NSNI}} \tag{4.11}$$

allows a much simpler treatment. One can set the ν_α produced in a NSNI to be a superposition of the mass eigenstates in another basis to that of the weak basis, this shall be discussed in more detail in the next section.

For a conventional beam using pion decay $\pi^+ \to \mu^+\nu_\mu$ it is assumed the lorentz structure is $(V-A)(V-A)$ for the NSNI term because of the kinematics of the pion decay, where the energy and helicity of the decaying particles μ and ν_μ are fixed. Therefore the simpler treatment in the next section using mass eigenstates is also valid.

4.3 $(V-A)(V-A)$ NSNI treatment

The weak interaction eigenstates $|\nu_\alpha^W\rangle$ were given previously as the superposition of the mass eigenstates $|\nu_i^m\rangle$ using the matrix U_{MNS} from Eq. (2.7). Here it is assumed that CP is conserved and so $U^W = (U_{\mathrm{MNS}})^{\delta=\beta_1=\beta_2=0} = (U^W)^*$ giving

$$|\nu_\alpha^W\rangle = \sum_i U_{\alpha i}^W |\nu_i^m\rangle \ . \tag{4.12}$$

As suggested in the last section, the effect of NSNI can be introduced by treating a neutrino produced as a superposition of the mass eigenstates in a new basis, which is defined as the source basis

$$|\nu_\alpha^s\rangle = \sum_i U_{\alpha i}^s |\nu_i^m\rangle \ . \tag{4.13}$$

Allowing for different NP at the detector than at the production process, the detector basis is defined as

$$|\nu_\alpha^d\rangle = \sum_i U_{\alpha i}^d |\nu_i^m\rangle \ . \tag{4.14}$$

It is also convenient to define small dimensionless quantities $\epsilon_{\alpha\beta}^s$ and $\epsilon_{\alpha\beta}^d$ as

$$\epsilon_{\alpha\beta}^s = \frac{(G_{\mathrm{NP}}^s)_{\alpha\beta}}{G_{\mathrm{F}}} \ ; \tag{4.15}$$

$$\epsilon_{\alpha\beta}^d = \frac{(G_{\mathrm{NP}}^d)_{\alpha\beta}}{G_{\mathrm{F}}} \ . \tag{4.16}$$

Where $\epsilon_{\alpha\beta}^{s,d} \ll 1$. For muon decay $\mu^- \to e^- \nu_\mu \bar{\nu}_e^s$ and detection via $\nu_\mu^d p \to \mu^- n$ [3] the source and detector eigenstates can then be written as

$$|\nu_e^s\rangle = \sum_i \underbrace{[U_{ei}^W + \epsilon_{e\mu}^s U_{\mu i}^W + \epsilon_{e\tau}^s U_{\tau i}^W]}_{U_{ei}^s} |\nu_i^m\rangle \ ; \tag{4.17}$$

$$|\nu_\mu^d\rangle = \sum_i \underbrace{[U_{\mu i}^W + \epsilon_{\mu e}^d U_{ei}^W + \epsilon_{\mu\tau}^d U_{\tau i}^W]}_{U_{\mu i}^d} |\nu_i^m\rangle \ , \tag{4.18}$$

with the contribution from flavour conserving NP assumed to be negligible. It is now easy to see that the new source and detector mixing matrices are not unitary. The eigenstates in Eqs. (4.17) and (4.18) can be used in the effective

[3] Also a $(V-A)(V-A)$ lorentz structure.

couplings of the interactions. For example in the production process the effective four fermion operator

$$\mathcal{L}^{eff}_{\mathrm{SM+NP}} = -2\sqrt{2}G_{\mathrm{F}}(\bar{\nu}_\mu \gamma_\rho L\mu)(\bar{e}\gamma^\rho L\nu^s_e) \tag{4.19}$$

would give all the relevant SM and NP interactions, with their corresponding couplings.

In the case of pion decay $\pi^+ \to \mu^+\nu_\mu$ for the production process and $\nu^d_e n \to e^- p$ for the detection process the effective four fermion operators of interest would be

$$-2\sqrt{2}(G^s_{\mathrm{NP}})_{\mu\alpha}(\bar{\nu}^W_\alpha \gamma_\rho L\mu)(\bar{u}\gamma^\rho Ld) \; ; \qquad \alpha = e,\ \tau \ , \tag{4.20}$$

$$-2\sqrt{2}(G^d_{\mathrm{NP}})_{e\alpha}(\bar{\nu}^W_\alpha \gamma_\rho Le)(\bar{u}\gamma^\rho Ld) \; ; \qquad \alpha = \mu,\ \tau \ , \tag{4.21}$$

as well as the normal SM flavour conserving effective four fermion operators at both the source and detector involving the coupling $G_{\mathrm{F}}\delta_{\alpha\beta}$, where $\alpha = \beta$.

For example in the production process the effective four fermion operator

$$\mathcal{L}^{eff}_{\mathrm{SM+NP}} = -2\sqrt{2}G_{\mathrm{F}}(\bar{\nu}^s_\mu \gamma_\rho L\mu)(\bar{u}\gamma^\rho Ld) \tag{4.22}$$

would give all the relevant SM and NP interactions, with their corresponding couplings.

4.4 Propagation in matter

It is now necessary to consider the propagation process. Recall in Section 2.3 that the Hamiltonian was modified for neutrino propagation in matter $H' \to H'_M$ where the mixing matrix was also transformed $U \to U_M$

$$\underbrace{UHU^\dagger}_{H'} + V^{eff}_{CC} = \underbrace{U_M H_M U^\dagger_M}_{H'_M} \ . \tag{4.23}$$

Taking the three state solution

$$H' = |p| + \begin{pmatrix} \frac{m_1^2}{2|p|} & & \\ & \frac{m_2^2}{2|p|} & \\ & & \frac{m_3^2}{2|p|} \end{pmatrix} \ . \tag{4.24}$$

Subtracting ($|p|+m_1^2$) as it will contribute the same to all eigenstates and setting $|p| \simeq E_\nu$ gives

$$(H'_M)_{\beta\alpha} = \frac{1}{2E_\nu}\left\{ U_{\beta i}\begin{pmatrix} 0 & & \\ & \delta m^2_{21} & \\ & & \delta m^2_{31} \end{pmatrix} U^\dagger_{i\alpha} + \begin{pmatrix} \bar{a} + a_{ee} & a_{e\mu} & a_{e\tau} \\ a_{e\mu}{}^* & a_{\mu\mu} & a_{\mu\tau} \\ a_{e\tau}{}^* & a_{\mu\tau}{}^* & a_{\tau\tau} \end{pmatrix}_{\beta\alpha} \right\} , \tag{4.25}$$

where $\bar{a}$ is the normal matter effect $\bar{a} = 2E_\nu V_{CC}^{eff} = 2\sqrt{2}G_F n_e E_\nu$ and $a_{\alpha\beta}$ is the extra matter effect due to NP defined by $a_{\alpha\beta} = 2\sqrt{2}\epsilon^m_{\alpha\beta} G_F n_e E_\nu$. It should be mentioned that the type of structure of the interaction here is irrelevant, since particles in ordinary matter are at rest and therefore the dependence on the chirality is averaged.

To obtain the total probability amplitude for a neutrino experiment involving NSNI one can write

$$P_{\nu^s_\alpha \to \nu^d_\beta} = |\langle \nu^d_\beta | e^{-i(H'_M)_{\beta\alpha}} | \nu^s_\alpha \rangle|^2 \ . \tag{4.26}$$

For a neutrino factory the transition $\nu^s_e \to \nu^d_\mu$ is the relevant probability to compute and one has

$$P_{\nu^s_e \to \nu^d_\mu} = |\langle \nu^d_\mu | \nu^s_e(t) \rangle|^2 \ , \tag{4.27}$$

where $|\nu^s_e(t)\rangle$ is the time evolved state of $|\nu^s_e(t=0)\rangle$ given by

$$|\nu^s_e(t)\rangle = \sum_i e^{-iH_M t} U^s_{ei} |\nu^m_i\rangle \ , \tag{4.28}$$

with

$$H_M = (U^\dagger_M)_{i\beta} (H'_M)_{\beta\alpha} (U_M)_{\alpha i} \ , \tag{4.29}$$

and $(H'_M)_{\beta\alpha}$ is given by Eq. (4.25). The state $|\nu^d_\mu\rangle$ is also given by

$$|\nu^d_\mu\rangle = U^d_{\mu i} |\nu^m_i\rangle \ . \tag{4.30}$$

Using the orthogonality of the mass eigenstates one obtains

$$P_{\nu^s_e \to \nu^d_\mu} = \left| \sum_i e^{-iH_M t} \left[U_{ei} U_{\mu i} + \epsilon^s_{e\mu} |U_{\mu i}|^2 + \epsilon^d_{\mu e} |U_{ei}|^2 + \epsilon^s_{e\tau} U_{\tau i} U_{\mu i} + \epsilon^d_{\mu\tau} U_{\mu i} U_{\tau i} \right] \right|^2 \ . \tag{4.31}$$

4.5 Upper bounds on the ϵ parameters

To have an idea of what contribution NSNI will give to oscillation probabilities it is necessary to find upper limits on the values for the effective couplings

$$(G_{\mathrm{NP}})^{s,d,m}_{\alpha\beta} = \epsilon^{s,d,m}_{\alpha\beta} G_F \ . \tag{4.32}$$

It is obvious that $\epsilon^{s,\ d,\ m}_{\alpha\beta} \ll 1$ due to dominance of the SM in experiments. However determining them beyond this is not a trivial matter, as four fermion operators for processes involving different particles may give different values for $\epsilon^{s,\ d,\ m}_{\alpha\beta}$. It is also necessary to distinguish between the different lorentz structures of the interactions. As mentioned previously a $(V-A)(V+A)$ structure for a charged current interaction cannot be treated simply, however the case is different for the neutral current involved in the propagation process

Z couplings	g_L^f	g_R^f
ν_e, ν_μ, ν_τ	$\frac{1}{2}$	0
e, μ, τ	$-\frac{1}{2} + \sin^2\theta_W$	$\sin^2\theta_W$
u, c, t	$\frac{1}{2} - \frac{2}{3}\sin^2\theta_W$	$-\frac{2}{3}\sin^2\theta_W$
d, s, b	$-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W$	$\frac{1}{3}\sin^2\theta_W$

Table 4.1: Z couplings to SM fermions.

of neutrinos in matter and one can treat the $(V-A)(V+A)$ NSNI interference effect in the same way as the $(V-A)(V-A)$ interference.

The SM interactions of neutrinos can be described by the effective lagrangian

$$\mathcal{L}_{eff}^{\mathrm{SM}} = -\underbrace{2\sqrt{2}G_F([\bar{\nu}_\beta\gamma_\rho L\ell_\beta][\bar{f}\gamma^\rho Lf'] + \mathrm{h.c.})}_{W^\pm} - \underbrace{2\sqrt{2}G_F\sum_{P,f,\beta} g_P^f[\bar{\nu}_\beta\gamma_\rho L\nu_\beta][\bar{f}\gamma^\rho Pf]}_{Z} \; , \tag{4.33}$$

where $P = \{L,\ R\}$, ℓ is a charged lepton, f is a lepton or quark, f' is its $SU(2)_L$ partner and g_P^f are the Z couplings given in Table (4.1). The NSNI extension to this is then

$$\mathcal{L}_{eff}^{\mathrm{NSNI}} = -\underbrace{\epsilon_{\alpha\beta}^{fL}2\sqrt{2}G_F(\bar{\nu}_\alpha\gamma_\rho L\ell_\beta)(\bar{f}\gamma^\rho Lf') + \mathrm{h.c.}}_{W^\pm} - \underbrace{\epsilon_{\alpha\beta}^{fP}2\sqrt{2}G_F(\bar{\nu}_\alpha\gamma_\rho L\nu_\beta)(\bar{f}\gamma^\rho Pf)}_{Z} \; . \tag{4.34}$$

As shown above it is necessary to introduce another two superscripts to the ϵ parameters. One superscript for the fermion f involved in the interaction and another P for the lorentz structure. Here it is assumed that the Z interactions are only present in matter during the propagation process and so the range of ϵ parameters are

$$\begin{array}{cc} Z & W^\pm \\ (\epsilon_{\alpha\beta}^{fP})^m & (\epsilon_{\alpha\beta}^{fL})^{s,\ d,\ m} \end{array} \; . \tag{4.35}$$

Depending on the production, propagation and detection process, bounds on only a few of these will need to be established for particular experiments.

It is easy to make an order of magnitude estimate for the bounds on the ϵ parameters if $SU(2)_L$ breaking effects are neglected, these effects are discussed in [55, 56]. Specific full high energy models at the loop level such as $\mu \to e\gamma$ and $\mu \to 3e$ give a good idea for bounds on $\epsilon_{\alpha\beta}^{eL}$ from the charged leptonic part of the neutrino mixing matrix U^ℓ, but extracting reliable estimates from loop processes in effective field theories is quite involved [58].

Some examples of the bounds on the ϵ parameters are those determined from the upper bounds on the lepton flavour violating decays $\mu^- \to e^- e^+ e^-$ and $\tau^- \to e^- e^+ e^-$ [9, 59] gives

$$\mathrm{BR}(\mu^- \to e^- e^+ e^-) < 1.0 \times 10^{-12} \ , \tag{4.36}$$

$$\mathrm{BR}(\tau^- \to e^- e^+ e^-) < 2.9 \times 10^{-6} \ . \tag{4.37}$$

After normalising these to the measured rates of the related lepton flavour conserving decays

$$\mathrm{BR}(\mu^- \to e^- \bar{\nu}_e \nu_\mu) \sim 1 \ , \tag{4.38}$$

$$\mathrm{BR}(\tau^- \to e^- \bar{\nu}_e \nu_\tau) \sim 0.18 \ , \tag{4.39}$$

allows bounds of [61]

$$\epsilon^{eL}_{\mu e} < 1.0 \times 10^{-6} \ , \tag{4.40}$$

$$\epsilon^{eL}_{\tau e} < 4.0 \times 10^{-3} \ . \tag{4.41}$$

In general it is assumed $\epsilon \leq 10^{-2}$ for most of the ϵ parameters. A more detailed analysis for the Z boson ϵ parameters can be found in [60] and for the $W^\pm$ bosons [61]

$$\begin{array}{lll} \epsilon^{eL}_{\mu e} \lesssim 1.0 \times 10^{-6} \ ; & \epsilon^{eL}_{\tau \mu} \lesssim 3.0 \times 10^{-3} \ ; & \epsilon^{eL}_{\tau e} \lesssim 4.0 \times 10^{-3} \ ; \\ \epsilon^{u,d\ L}_{\mu e} \lesssim 1.0 \times 10^{-5} \ ; & \epsilon^{u,d\ L}_{\tau \mu} \lesssim 1.0 \times 10^{-2} \ ; & \epsilon^{u,d\ L}_{\tau e} \lesssim 1.0 \times 10^{-2} \ . \end{array} \tag{4.42}$$

It is estimated [62] that within 5 years of operation at a neutrino factory with $E_\nu = 50$ GeV a sensitivity to $\epsilon \sim \mathcal{O}(10^{-4})$ could be reached.

4.6 New Physics sources of NSNI

4.6.1 Effective four-lepton interactions

If it will be possible to measure the ϵ parameters to within $\sim \mathcal{O}(10^{-4})$ once a neutrino factory has been built, then it is necessary to investigate their possible sources from NP. There are many different extensions to the SM that could produce such NP, so the most general case is looked at first.

Flavour violating four-lepton couplings with a $(V-A)(V-A)$ lorentz structure can be put into two categories depending on their transformations under the $SU(2)_L$ gauge group. There is the singlet type

$$\begin{aligned} & h_{\alpha\beta\gamma\delta} (\bar{l}_\alpha C \bar{l}_\beta)(l_\gamma C^\dagger l_\delta) \\ &= -\frac{1}{2} h_{\alpha\beta\gamma\delta} (\bar{l}_\alpha \gamma^\mu l_\delta)(\bar{l}_\beta \gamma_\mu l_\gamma) \\ &= \frac{1}{2} h_{\alpha\beta\gamma\delta} h_{\alpha\beta\gamma\delta} (\bar{e}_{\alpha L} C \bar{\nu}_{\beta L} - \bar{\nu}_{\alpha L} C \bar{e}_{\beta L})(e_{\gamma L} C^\dagger \nu_{\delta L} - \nu_{\gamma L} C^\dagger e_{\delta L}) \ , \end{aligned} \tag{4.43}$$

and the triplet type,

$$\begin{aligned} & g_{\alpha\beta\gamma\delta}(\bar{l}_\alpha \tau^a C \bar{l}_\beta)(l_\gamma C^\dagger \tau^a l_\delta) \\ &= -\frac{1}{2} g_{\alpha\beta\gamma\delta}(\bar{l}_\alpha \tau^a \gamma^\mu l_\delta)(\bar{l}_\beta \tau^a \gamma_\mu l_\gamma) \\ &= g_{\alpha\beta\gamma\delta} \left\{ (\bar{e}_{\alpha L} C \bar{\nu}_{\beta L} + \bar{\nu}_{\alpha L} C \bar{e}_{\beta L})(e_{\gamma L} C^\dagger \nu_{\delta L} + \nu_{\gamma L} C^\dagger e_{\delta L}) \right. \\ &\quad \left. + 2(\bar{\nu}_{\alpha L} C \bar{\nu}_{\beta L})(\nu_{\gamma L} C^\dagger \nu_{\delta L}) + 2(\bar{e}_{\alpha L} C \bar{e}_{\beta L})(e_{\gamma L} C^\dagger e_{\delta L}) \right\} \ . \end{aligned} \tag{4.44}$$

where ℓ_α is a lepton doublet, τ^a is a Pauli matrix and the coupling constants satisfy $h_{\alpha\beta\gamma\delta} = -h_{\beta\alpha\gamma\delta} = -h_{\alpha\beta\delta\gamma}$ and $g_{\alpha\beta\gamma\delta} = g_{\beta\alpha\gamma\delta} = g_{\rho\sigma\delta\gamma}$ because of $SU(2)_L$ invariance. Both of these interactions could be induced by the exchange of a vector or scalar particle. For example Eq. (4.43) allows a coupling of the form

$$\bar{l} C \bar{l} \phi_{\mathrm{S}} + \mathrm{h.c.} \ , \tag{4.45}$$

with ϕ_{S} as a $SU(2)_L$ singlet and Eq. (4.44) allows a coupling of the form

$$\bar{l} \tau^a C \bar{l} \phi^a_{\mathrm{T}} + \mathrm{h.c.} \ , \tag{4.46}$$

with ϕ^a_{T} as a $SU(2)_L$ triplet. There is also a flavour violating four-lepton coupling for the $(V-A)(V+A)$ lorentz structure giving a scalar particle with an exchange belonging to a $SU(2)_L$ doublet however as mentioned before this structure is highly suppressed and so for this particular discussion of NP it shall not be mentioned.

4.6.2 Supersymmetric models I

Supersymmetric models with right-handed neutrinos are one source for the triplet type couplings described in Section 4.6.1. Here gravity induced supersymmetry breaking is utilised, using the renormalisation effect and large flavour violating slepton masses are introduced [64]. These masses allow one loop box diagrams to be made using superpartner propagators, giving an effective $(V-A)(V-A)$ NSNI. For a neutrino factory muon decay is the production process for neutrinos and Fig. (4.6-(1)) shows one such allowed supersymmetric effective interaction. Other one loop diagrams for effective interactions can be introduced using squark propagators for the detection and production processes involving quarks. Fig. (4.6-(2)) shows a one loop box diagram for such a process.

Both Fig. (4.6-(1)) and Fig. (4.6-(2)) will affect neutrino factory and conventional neutrino beam experiments differently. Therefore by comparing data from each it might be possible to obtain information on the masses of the scalar exchange particles. If slepton masses are much lighter than squark masses then interactions of the type in Fig. (4.6-(1)) will contribute less to NSNI and there should be a significant difference in probabilities measured at each type of experiment.

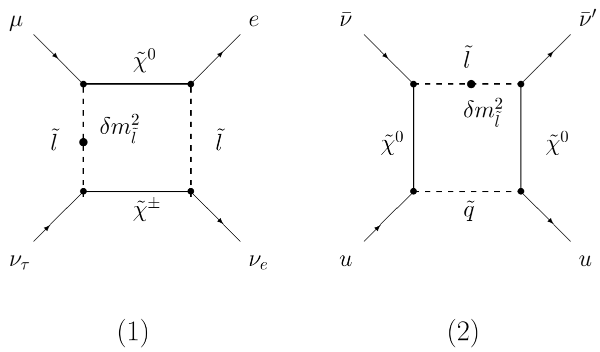

Figure 4.6: [54] (1) A supersymmetric one loop box diagram for a possible NSNI contribution to muon decay in a neutrino factory. (2) A supersymmetric one loop box diagram for a possible NSNI contribution to both pion decay and the detection process. $\delta m^2_{\tilde{l}}$ denotes an off-diagonal element of the lepton flavour violating slepton mass matrix.

4.6.3 Supersymmetric models II

Supersymmetric models with R-Parity violating terms are one source for the singlet type couplings described in Section 4.6.1. The most general superpotential which breaks R-Parity is

$$W_{\rm RPV} = \lambda_{\alpha\beta\gamma} L_\alpha L_\beta E^c_\gamma + \lambda'_{\alpha\beta\gamma} L_\alpha Q_\beta D^c_\gamma + \lambda''_{\alpha\beta\gamma} U^c_\alpha D^c_\beta D^c_\gamma + \mu'_\alpha L_\alpha H_{U\alpha} \ , \quad (4.47)$$

where

$$\lambda_{\alpha\beta\gamma} = -\lambda_{\beta\alpha\gamma}, \ \lambda''_{\alpha\beta\gamma} = -\lambda''_{\alpha\gamma\beta} \ , \quad (4.48)$$

with L_α , E^c_α, Q_α, U_α, D^c_α and $H_{U\alpha}$ as the superfields corresponding to the lepton doublets, the right-handed charged leptons, the quark doublets, the right-handed up-type-quarks, the down-type quarks and the up-type Higgs doublet respectively. To induce neutrinos with small masses, the first and second terms must be included in the superpotential.

Taking the first term gives interactions described by the lagrangian

$$\mathcal{L} = \lambda_{\alpha\beta\gamma}(\bar{\nu}^c_{\alpha L} e_{\beta L} \tilde{e}^*_{\gamma R} + \bar{e}_{\gamma R} \nu_{\alpha L} \tilde{e}_{\beta L} + \bar{e}_{\gamma R} e_{\beta L} \tilde{\nu}_{\alpha L} - (\alpha \leftrightarrow \beta)) + \text{h.c.} \ . \quad (4.49)$$

Here sleptons can interact with the charged leptons and neutrinos and small neutrino masses with lepton mixing is subsequently introduced radiatively, as shown in Fig. (4.7-(1)). These Types of NSNI could also be distinguished using a neutrino factory and conventional beam experiment as they would only affect muon decay and perhaps interactions while travelling through matter.

Taking the second term gives interactions described by the lagrangian

$$\begin{aligned}\mathcal{L} = {} & \lambda'_{\alpha\beta\gamma} V^{CKM}_{\beta\delta} (\bar{\nu}^c_{\alpha L} d_{\delta L} \tilde{d}^*_{\gamma R} + \bar{d}_{\gamma R} \nu_{\alpha L} \tilde{d}_{\delta L} + \bar{d}_{\gamma R} d_{\delta L} \tilde{\nu}_{\alpha L}) \\ & + \lambda'_{\alpha\beta\gamma} (\bar{u}^c_{\beta L} e_{\alpha L} \tilde{d}^*_{\gamma R} + \bar{d}_{\gamma R} e_{\alpha L} \tilde{u}_{\beta L} + \bar{d}_{\gamma R} u_{\beta L} \tilde{e}_{\alpha L}) + \text{h.c.} \ , \quad (4.50)\end{aligned}$$

where V^{CKM} is the mixing matrix for the quark sector. These interactions are generated in a similar way to those of the first term, allowing lepton flavour

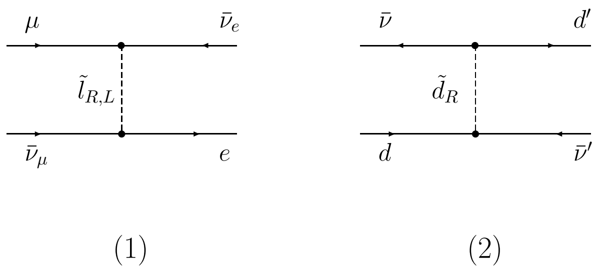

Figure 4.7: [54] (1) Tree diagram showing a possible NSNI contribution to muon decay. (2) Tree diagram showing a possible contribution to pion decay, matter and detection processes.

violation as shown in Fig. (4.7-(2)). These types of interactions could also be distinguished by comparing the neutrino factory and conventional beam experiments.

Chapter 5

Non-Decoupling of Heavy Neutrinos

5.1 Overview

The number (n_L) of active light neutrinos in the SM is now widely accepted as being equal to three due to the measurement of the Z decay width (see Chapter 1). However there is the possibility that there are more neutrino species with smaller couplings to the Z boson than that of the three calculated. An extra number of light sterile neutrinos (n_S) could be present in nature[1] along with a number of heavy sterile neutrinos (n_R) as indicated by the See-saw mechanism. The equations derived throughout this chapter are taken from a paper by Bekman et al. (2002) [65].

U_ν is introduced to denote the full neutrino unitary mixing matrix written as

$$U_\nu = \begin{pmatrix} \mathcal{U} & \mathcal{V} \\ \mathcal{V}' & \mathcal{U}' \end{pmatrix} , \tag{5.1}$$

with

- A submatrix $\mathcal{U}$ with dimension $(3+n_S)\times(3+n_S)$ involved in the mixing of the 3+n_S light neutrino states.
- A submatrix $\mathcal{U}'$ with dimension $n_R \times n_R$ involved in the mixing of the n_R heavy neutrino states.
- Submatrices $\mathcal{V}$ and $\mathcal{V}'$ with dimension $(3+n_S)\times n_R$ and $n_R \times (3+n_S)$ respectively, involved in the mixing of the 3+n_S light neutrino states with the n_R heavy neutrino states.

[1] At least one is necessary to understand the LSND oscillation experiment.

In the case of the See-saw mechanism, the light-heavy neutrino mixing matrix $\mathcal{V}$ is quite negligible and therefore the matrix $\mathcal{U}$ is very close to unitarity. However if the See-saw mechanism is not in operation or heavy neutrino mixing is taken into account (non-decoupling of heavy neutrinos), then $\mathcal{U}$ becomes more non-unitary as the elements of $\mathcal{V}$ become more substantial. Universality constrains the diagonal elements of

$$c_{\alpha\beta} \equiv (\mathcal{V}\mathcal{V}^\dagger)_{\alpha\beta} = \delta_{\alpha\beta} - (\mathcal{U}\mathcal{U}^\dagger)_{\alpha\beta} \ , \tag{5.2}$$

while the off-diagonal elements are constrained in the same way as the ϵ parameters were in the last chapter. Here the non-observation of lepton violating processes such as $\mu \to e\gamma$, $\mu \to 3e$ etc. gives [63]

$$\begin{gathered} c_{ee} < 0.0054 \ , \qquad c_{\mu\mu} < 0.0096 \ , \qquad c_{\tau\tau} < 0.0016 \ , \\ |c_{e\mu}| < 0.0001 \ , \qquad |c_{e\tau}| < 0.01 \ . \end{gathered} \tag{5.3}$$

The submatrix $\mathcal{U}$ for mixing among the light neutrinos is therefore **not** unitary when heavy neutrinos do not decouple

$$(\mathcal{U}\mathcal{U}^\dagger)_{\alpha\beta} = \delta_{\alpha\beta} - c_{\alpha\beta} \ . \tag{5.4}$$

5.2 The interaction hamiltonian in matter

Neutrino interactions within the SM are described by the lagrangians

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} \sum_{\ell=e,\mu,\tau} \sum_{i=1}^{n} \bar{\ell}\gamma^\mu (1-\gamma_5)(U_\nu)_{\ell i}\nu_i W_\mu^- + \text{h.c} \ , \tag{5.5}$$

and

$$\begin{aligned} \mathcal{L}_{NC} &= -\frac{g}{2\cos\theta_W} \left\{ \sum_{i,j=1}^{n} \bar{\nu}_i \gamma^\mu (1-\gamma_5)\, \Omega_{ij} \nu_j Z_\mu \right. \\ &\left. + 2 \sum_{f=e,p,n} \bar{f}\gamma^\mu \left[T_{3f}(1-\gamma_5) - 2Q_f \sin^2\theta_W\right] f Z_\mu \right\} \ , \end{aligned} \tag{5.6}$$

where n is the number of light and heavy neutrinos ($n = 3 + n_s + n_R$) and $\Omega_{ij} = \sum_{\alpha=e,\mu,\tau} (U_\nu)^*_{\alpha i}(U_\nu)_{\alpha j}$. These interactions are shown in Fig. (5.1). The effective interaction hamiltonian for light neutrinos in matter is then given as

$$\mathcal{H}^{eff}_{NC+CC}(x) = \frac{G_F}{\sqrt{2}} \sum_{i,k=1}^{3+n_s} \sum_{a=V,A} \left[\bar{\nu}_k \Gamma_a \nu_i\right] \left[\bar{f}\Gamma^a \left(g^{ki}_{fa} + \bar{g}^{ki}_{fa}\gamma_5\right) f\right] \ , \tag{5.7}$$

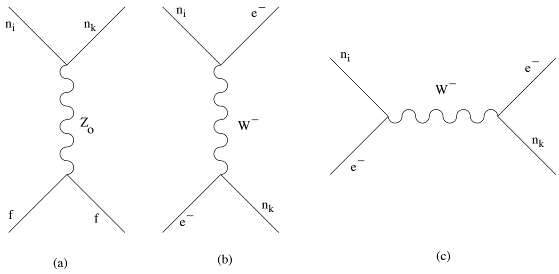

Figure 5.1: [65] Feynman diagrams for possible SM interactions of neutrinos in matter. All contribute toward neutrino-electron scattering but only (a) contributes toward neutrino-nucleon scattering.

where $\Gamma_{V(A)} = \gamma_\mu(\gamma_\mu\gamma_5)$ and the relevant g couplings for this discussion are

$$\begin{aligned}
g_{eV}^{ki} &= -\bar{g}_{eA}^{ki} = \mathcal{U}_{ek}^{*}\mathcal{U}_{ei} + \rho\Omega_{ki}\left(-\frac{1}{2} + 2\sin^2\theta_W\right) \ , \\
\bar{g}_{eV}^{ki} &= -g_{eA}^{ki} = -\mathcal{U}_{ek}^{*}\mathcal{U}_{ei} + \frac{1}{2}\rho\Omega_{ki} \ , \\
g_{fV}^{ki} &= -\bar{g}_{fA}^{ki} = \rho\Omega_{ki}\left(T_{3f} - 2Q_f\sin^2\theta_W\right) , \\
\bar{g}_{fV}^{ki} &= -g_{eA}^{ki} = -\rho\Omega_{ki}T_{3f} \ ,
\end{aligned} \tag{5.8}$$

along with

$$\rho = \frac{M_W^2}{M_Z^2\cos^2\theta_W} \simeq 1, \ T_{3p} = -T_{3n} = 1/2, \ Q_p = 1, \ Q_n = 0 \ . \tag{5.9}$$

The hamiltonian of Eq. (5.7) can be given globally as

$$\mathcal{H}_{NC+CC}^{eff}(x) = \sum_{i,k=1}^{3+n_s} \bar{\nu}_k V_{ki} \nu_i \ , \tag{5.10}$$

where V_{ki} is the effective potential

$$V_{ki} = \sum_f V_{ki}^f = \sum_f \sum_a \Gamma^a \left(V_a^f\right)_{ki} \ , \tag{5.11}$$

with

$$\left(V_a^f\right)_{ki} = \frac{G_F}{\sqrt{2}} \sum_{\vec{\lambda}} \int \frac{d^3p}{(2\pi)^3} \rho_f(\vec{p},\vec{\lambda}) \left(M_a^f\right)_{ki} \ . \tag{5.12}$$

Here $\left(M_a^f\right)_{ki}$ denotes part of the matrix element for the scattering amplitude $\nu_i + f \to \nu_k + f$ when all the particles' momenta and spins are left unchanged

$$\left(M_a^f\right)_{ki} = \langle f,\vec{p},\vec{\lambda}|\bar{f}\Gamma_a\left(g_{fa}^{ki} + \bar{g}_{fa}^{ki}\gamma_5\right) f|f,\vec{p},\vec{\lambda}\rangle \ . \tag{5.13}$$

The distribution function for particles of spin $\vec{\lambda}$ and momentum $\vec{p}$ is given as $\rho_f(\vec{p},\vec{\lambda})$ in Eq. (5.12) and can be normalised so that the number of fermions per unit volume N_f is

$$N_f \equiv \sum_{\vec{\lambda}} \int \frac{d^3p}{(2\pi)^3} \rho_f(\vec{p},\vec{\lambda}) \ . \tag{5.14}$$

The amplitude $\left(M_a^f\right)_{ki}$ for one fermion in $V=1$ is then

$$\left(M_V^f\right)_{ki}^{\mu} = -\left(M_A^f\right)_{ki}^{\mu} = g_{fV}^{ki}\left(\frac{p^\mu}{E_f}\right) + m_f \bar{g}_{fV}^{ki}\left(\frac{S_f^\mu}{E_f}\right) \ , \tag{5.15}$$

with E_f, m_f and $S_f^\mu = \frac{1}{m_f}\left(\vec{p}\vec{\lambda}, \vec{\lambda}m_f + \frac{\vec{p}(\vec{p}\vec{\lambda})}{m_f+E_f}\right)$ denoting the fermion's energy, mass and spin respectively. The effective potential V_{ki}^f in Eq. (5.11) can now be written explicitly as

$$V_{ki}^f = \left(A_f^\mu\right)_{ki}\gamma_\mu P_L \ , \tag{5.16}$$

where

$$\left(A_f^\mu\right)_{ki} = \sqrt{2}G_F N_f\left[g_{fV}^{ki}\langle\frac{p_f^\mu}{E_f}\rangle + m_f\bar{g}_{fV}^{ki}\langle\frac{S_f^\mu}{E_f}\rangle\right] \ , \tag{5.17}$$

and $<\chi>$ is defined as

$$<\chi>=\frac{1}{N_f}\sum_{\vec{\lambda}}\int\frac{d^3p}{(2\pi)^3}\rho_f(\vec{p},\vec{\lambda})\chi(\vec{p},\vec{\lambda}) \ . \tag{5.18}$$

If the neutrinos are light and relativistic, their propagation can be described by

$$i\frac{d}{dt}\Psi_k(\vec{k},t) = \sum_{i=1}^{3+n_s} H_{ki}^{eff}(t)\Psi_i(\vec{k},t) \ , \tag{5.19}$$

where $\Psi_j(\vec{k},t)$ is the neutrino (antineutrino) state $|\Psi(t)\rangle$ with momentum $\vec{k}$ and helicity $\lambda=-1(+1)$ given by $\Psi_j = \langle\nu_j|\Psi(t)\rangle$ where $|\nu_j\rangle$ are the mass eigenstates. Assuming all the neutrinos have the same momentum but different masses, the effective hamiltonian in the mass basis becomes

$$H_{ki}^{eff} = \left(k + \frac{m_i^2}{2k}\right)\delta_{ki} + H_{ki}^{int} \ , \tag{5.20}$$

where

$$\begin{aligned} H_{ki}^{int} &= \langle\nu_k|\int_{V=1} d^3x\, \mathcal{H}_{NC+CC}^{eff}(x)\,|\nu_i\rangle \text{ and using Eq. (5.10)} \\ &= \begin{cases} A_{ki}^\mu \bar{u}_k\gamma_\mu P_L u_i & \text{Dirac neutrinos} \\ -(A_{ki}^\mu)^*\bar{u}_k\gamma_\mu P_R u_i & \text{Dirac antineutrinos} \\ A_{ki}^\mu\bar{u}_k\gamma_\mu P_L u_i - (A_{ki}^\mu)^*\bar{u}_k\gamma_\mu P_R u_i & \text{Majorana neutrinos} \end{cases} \end{aligned} \tag{5.21}$$

with $A_{ki}^\mu = \sum_f\left(A_f^\mu\right)_{ki}$. In the relativistic limit and assuming $\vec{k}=\vec{k}_i=\vec{k}_f$, H_{ki}^{int} is then

$$H_{ki}^{int} = \begin{cases} A_{ki}^0 - \frac{\vec{k}}{|\vec{k}|}\vec{A}_{ki} & \text{Dirac and Majorana neutrinos with } \lambda=-1, \\ -(A_{ki}^0)^* + \frac{\vec{k}}{|\vec{k}|}(\vec{A}_{ki})^* & \text{Dirac antineutrinos and Majorana neutrinos with } \lambda=+1. \end{cases} \tag{5.22}$$

This is the most general form for the interaction hamiltonian for any number of light relativistic neutrinos propagating in matter.

5.3 Decoupled heavy neutrinos

If a medium that is isotropic, unpolarised and electrically neutral like the earth is considered ($N_e = N_p \neq N_n$) then using Eq. (5.17) and Eq. (5.8), allows Eq. (5.22) to be reduced to

$$H^{int}_{ki} = \sqrt{2} G_F \left[N_e \mathcal{U}^*_{ek} \mathcal{U}_{ei} - \frac{1}{2} N_n \Omega_{ki} \right] \ . \tag{5.23}$$

Here H^{int}_{ki} will **always** have $(3+n_s) \times (3+n_s)$ dimensions even if there are heavy neutrinos present.

In the case where no heavy neutrinos are present, but light sterile neutrinos are still possible, the mixing between the weak interaction eigenstates is given by

$$|\nu_\alpha\rangle = \sum_{i=1}^{n} (U^*_\nu)_{\alpha i} |\nu_i\rangle = \sum_{i=1}^{3+n_s} \mathcal{U}^*_{\alpha i} |\nu_i\rangle \ , \tag{5.24}$$

with $\mathcal{U}^*_{\alpha i}$ as a unitary matrix. The effective hamiltonian for the weak interaction eigenstates in matter can be found by substituting Eq. (5.23) into Eq. (5.20), transforming this with $\mathcal{U}$ and removing all general phases. One then obtains

$$H_{\alpha\beta} = \left(\mathcal{U} \frac{\Delta m^2}{2E} \mathcal{U}^\dagger \right)_{\alpha\beta} + \sqrt{2} G_F \left[N_e \delta_{\alpha e} \delta_{\beta e} + \frac{1}{2} N_n \delta_{\alpha s} \delta_{\beta s} \right] \ . \tag{5.25}$$

where $\alpha, \beta = \{e,\ \mu,\ \tau,\ s_1,\ s_2 \cdots\}$, $s = \{s_1,\ s_2 \cdots\}$ and s_i are the light sterile neutrinos. If no light sterile neutrinos are present, then one obtains the well known hamiltonian from Chapter 2

$$H_{\alpha\beta} = \left(\mathcal{U} \frac{\Delta m^2}{2E} \mathcal{U}^\dagger \right)_{\alpha\beta} + \sqrt{2} G_F N_e \delta_{\alpha e} \delta_{\beta e} \ , \tag{5.26}$$

where $\alpha, \beta = \{e,\ \mu,\ \tau\}$.

5.4 Non-decoupled heavy neutrinos

If there is more than one heavy neutrino present ($n_R \geq 1$), the full hamiltonian H^{eff}_{ki} of Eq. (5.20) will need to be expanded to $(3+n_s+n_R) \times (3+n_s+n_R)$ dimensions. This can be achieved by adding zeros, assuming that the heavy neutrinos cannot be produced or detected due to energy and momentum conservation.

$$H^{eff}_{ki} \rightarrow \begin{cases} H^{eff}_{ki} \text{as given by Eqs. 5.20, 5.23} & \text{if both } k, i \leq (3+n_s) \ , \\ 0 & \text{if any of } k, i > (3+n_s) \ . \end{cases} \tag{5.27}$$

Taking heavy neutrino mixing into account means that Eq. (5.24) now becomes

$$|\nu_\alpha\rangle = \sum_{i=1}^{n} (U^*_\nu)_{\alpha i} |\nu_i\rangle = \sum_{i=1}^{3+n_s} \mathcal{U}^*_{\alpha i} |\nu_i\rangle + \sum_{i=3+n_s+1}^{3+n_s+n_R} \mathcal{V}^*_{\alpha i} |\nu_i\rangle \ . \tag{5.28}$$

Now both $\mathcal{U}^*_{\alpha i}$ and $\mathcal{V}^*_{\alpha i}$ are non-unitary. There are two cases for the effective hamiltonian $H^{eff}_{ki} \to H_{\alpha\beta}$ in the weak interaction basis. The first corresponds to having no light sterile, but some heavy neutrinos ($n_s = 0 \to n = 3 + n_R$)

$$\begin{aligned} H_{\alpha\beta} &= \left(\mathcal{U}\frac{\Delta m^2}{2E}\mathcal{U}^\dagger\right)_{\alpha\beta} \\ &\quad + \sqrt{2}G_F \Bigg[N_e(\delta_{\alpha e}\delta_{\beta e} - c_{\alpha\beta}(\delta_{\alpha e} + \delta_{\beta e}) + c_{\alpha e}c_{e\beta}) \\ &\quad + \frac{1}{2}N_n(2c_{\alpha\beta} - \sum_{\gamma=e,\mu,\tau} c_{\alpha\gamma}c_{\gamma\beta}) \Bigg] \ , \end{aligned} \tag{5.29}$$

where $\alpha, \beta = \{e,\ \mu,\ \tau\}$. The second corresponds to having light sterile neutrinos and heavy neutrinos ($n = 3 + n_s + n_R$)

$$\begin{aligned} H_{\alpha\beta} &= \left(\mathcal{U}\frac{\Delta m^2}{2E}\mathcal{U}^\dagger\right)_{\alpha\beta} \\ &\quad + \sqrt{2}G_F \Bigg[N_e \left\{\delta_{\alpha e}\delta_{\beta e} - c_{\alpha\beta}(\delta_{\alpha e} + \delta_{\beta e}) + c_{e\alpha}c_{e\beta}\right\} \\ &\quad + \frac{1}{2}N_n \left\{ \delta_{\alpha s}\delta_{\beta s} + c_{\alpha\beta}(2 - \delta_{\alpha s} - \delta_{\beta s}) - \sum_{\gamma=e,\mu,\tau} c_{\alpha\gamma}c_{\gamma\beta} \right\} \Bigg] \ , \end{aligned} \tag{5.30}$$

where $\alpha, \beta = \{e,\ \mu,\ \tau,\ s_1,\ s_2 \cdots\}$, $s = \{s_1,\ s_2 \cdots\}$ and s_i are the light sterile neutrinos. Both of these hamiltonians are hermitian and either of them would reside in the light neutrino subspace only ie. the upper left part of the full hamiltonian which is a $(3 + n_s + n_R) \times (3 + n_s + n_R)$ matrix represented by

$$H_{\alpha\beta} \equiv \begin{pmatrix} \mathcal{U}H^{eff}_{ki}\mathcal{U}^\dagger & \mathcal{U}H^{eff}_{ki}\mathcal{V}'^\dagger \\ \mathcal{V}'H^{eff}_{ki}\mathcal{U}^\dagger & \mathcal{V}'H^{eff}_{ki}\mathcal{V}'^\dagger \end{pmatrix}_{\alpha\beta} \ . \tag{5.31}$$

Here H^{eff}_{ki} is the $(3+n_s)\times(3+n_s)$ matrix given by Eqs. (5.20), (5.23) and $\alpha, \beta = \{e,\ \mu,\ \tau,\ s_1,\ s_2 \cdots r_1,\ r_2 \cdots\}$ where r_i are the heavy neutrinos. Again assuming that no heavy neutrino eigenstates can actually be produced or detected but are still present, one can introduce the normalised states $|\tilde{\nu}_\alpha\rangle$ corresponding to the **real** neutrinos produced/detected as

$$|\tilde{\nu}_\alpha\rangle = \lambda_\alpha^{-1} \sum_{i=1}^{3+n_s} \mathcal{U}^*_{\alpha i} |\nu_i\rangle = \sum_{i=1}^{3+n_s} \tilde{\mathcal{U}}^*_{\alpha i} |\nu_i\rangle \ . \tag{5.32}$$

Here $\lambda_\alpha = \sqrt{\sum_{i=1}^{3+n_s} |\mathcal{U}_{\alpha i}|^2} = \sqrt{1-c_{\alpha\alpha}}$ and $\tilde{\mathcal{U}}_{\alpha i} = \lambda_\alpha^{-1}\mathcal{U}_{\alpha i}$ is also not unitary. It is quite evident that these states are not orthogonal states like those of the weak interaction.

$$\langle\tilde{\nu}_\alpha|\tilde{\nu}_\beta\rangle \neq 0, \quad \alpha \neq \beta \; . \tag{5.33}$$

This is a very important point, as it means that a neutrino state $|\tilde{\nu}_e\rangle$ produced by an electron can also be produced by a muon or tau particle. The evolution equation for **real** neutrino states then becomes

$$i\frac{d}{dt}\langle\tilde{\nu}_\alpha|\Psi(t)\rangle = \sum_\beta \tilde{H}_{\alpha\beta}\langle\tilde{\nu}_\beta|\Psi(t)\rangle \; . \tag{5.34}$$

with $\tilde{H}_{\alpha\beta}$ as a non-hermitian hamiltonian

$$\begin{aligned}
\tilde{H}_{\alpha\beta} &= \tilde{\mathcal{U}} H_{\alpha\beta} \tilde{\mathcal{U}}^{-1} \\
&= \frac{1}{2E}\tilde{\mathcal{U}} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \delta m_{21}^2 & 0 & 0 & \cdots \\ 0 & 0 & \delta m_{31}^2 & 0 & \cdots \\ 0 & 0 & 0 & \delta m_{41}^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix} \tilde{\mathcal{U}}^{-1} \\
&\quad + \sqrt{2}G_F \tilde{\mathcal{U}}\tilde{\mathcal{U}}^\dagger \lambda^2 \begin{pmatrix} (N_e - \frac{N_n}{2}) & 0 & 0 & 0 & \cdots \\ 0 & -\frac{N_n}{2} & 0 & 0 & \cdots \\ 0 & 0 & -\frac{N_n}{2} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix} .
\end{aligned} \tag{5.35}$$

However it is always more convenient to work in the mass basis where Eq. (5.34) becomes

$$i\frac{d}{dt}\langle\nu_k|\tilde{\nu}_\alpha(t)\rangle = \sum_{i=1}^{3+n_s} H_{ki}\langle\nu_i|\tilde{\nu}_\alpha(t)\rangle \; , \tag{5.36}$$

with

$$H_{ki} = \frac{1}{2E}\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \delta m_{21}^2 & 0 & 0 & \cdots \\ 0 & 0 & \delta m_{31}^2 & 0 & \cdots \\ 0 & 0 & 0 & \delta m_{41}^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}$$
$$+\sqrt{2}G_F\mathcal{U}^\dagger \begin{pmatrix} (N_e - \frac{N_n}{2}) & 0 & 0 & 0 & \cdots \\ 0 & -\frac{N_n}{2} & 0 & 0 & \cdots \\ 0 & 0 & -\frac{N_n}{2} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}\mathcal{U} \ , \tag{5.37}$$

and the initial condition

$$\langle \nu_k | \tilde{\nu}_\alpha(0)\rangle = \lambda_\alpha^{-1}\mathcal{U}_{\alpha k}^* \ . \tag{5.38}$$

5.5 An example of three light active neutrinos and one non-decoupled heavy neutrino

Here the theory of the last sections is applied to the case where one non-decoupled heavy neutrino is present in a three light active neutrino state solution. It is also assumed that the heavy neutrino cannot be produced in any interactions due to its large mass.

The mixing between the light active neutrinos and the heavy sterile neutrino is given by Eq. (5.28), where the total mixing matrix U_ν is unitary, but $\mathcal{U}$ and $\mathcal{V}$ are not. $\mathcal{V}$ can be parameterised by

$$\mathcal{V} = \begin{pmatrix} \epsilon_e \\ e^{-i\chi_1}\epsilon_\mu \\ e^{-i\chi_2}\epsilon_\tau \end{pmatrix} \ , \tag{5.39}$$

where the real parameters $\epsilon_e, \epsilon_\mu, \epsilon_\tau$, and real phases χ_1, χ_2 (responsible for additional CP and T asymmetry effects in the mixing process) have been introduced. With $\lambda_\alpha = \sqrt{1-\epsilon_\alpha^2} \approx 1 - \epsilon_\alpha^2/2$, the matrix $\mathcal{U}$ to the order of $\mathcal{O}(\epsilon_\alpha^2)$ has the form

$$\mathcal{U} = \begin{pmatrix} U_{e1}\lambda_e, & U_{e2}\lambda_e, & U_{e3}\lambda_e \\ \\ U_{\mu 1}\lambda_\mu - U_{e1}e^{-i\chi_1}\epsilon_e\epsilon_\mu, & U_{\mu 2}\lambda_\mu - U_{e2}e^{-i\chi_1}\epsilon_e\epsilon_\mu, & U_{\mu 3}\lambda_\mu - U_{e3}e^{-i\chi_1}\epsilon_e\epsilon_\mu \\ \\ U_{\tau 1}\lambda_\tau - U_{e1}e^{-i\chi_1}\epsilon_e\epsilon_\tau & U_{\tau 2}\lambda_\tau - U_{e2}e^{-i\chi_1}\epsilon_e\epsilon_\tau & U_{\tau 3}\lambda_\tau - U_{e3}e^{-i\chi_1}\epsilon_e\epsilon_\tau \\ -U_{\mu 1}e^{i(\chi_1-\chi_2)}\epsilon_\mu\epsilon_\tau, & -U_{\mu 2}e^{i(\chi_1-\chi_2)}\epsilon_\mu\epsilon_\tau, & -U_{\mu 3}e^{i(\chi_1-\chi_2)}\epsilon_\mu\epsilon_\tau \end{pmatrix}, \tag{5.40}$$

with $U_{\alpha i}$ as the elements of the normal unitary U_{MNS} matrix for three light active neutrinos (see chapter 2). The ϵ parameters are constrained by the experimental data as described in section 5.1 where one has

$$\epsilon_e^2 < 0.0054 \ , \qquad \epsilon_\mu^2 < 0.0096 \ , \qquad \epsilon_\tau^2 < 0.016 \ ,$$
$$\epsilon_e\epsilon_\mu < 0.0001 \ , \qquad \epsilon_\mu\epsilon_\tau < 0.01 \ . \tag{5.41}$$

In this discussion it is assumed that CP is conserved and so the phases χ_1 and χ_2 shall be neglected from here on. The effective hamiltonian from Eq. (5.20) is written as

$$H_{ki}^{eff} = \frac{1}{2E}\left(m_i^2\delta_{ki} + 2\sqrt{2}G_F E\left(N_e\mathcal{U}_{ek}^*\mathcal{U}_{ei} - \frac{1}{2}N_n\Omega_{ki}\right)\right) \ , \tag{5.42}$$

which can be diagonalised to give the effective neutrino masses in matter using the unitary matrix $\tilde{W}$

$$H^{eff} = \frac{1}{2E}\tilde{W}^\dagger diag(\tilde{m}_i^2)\tilde{W} \ . \tag{5.43}$$

The evolution equation then becomes

$$i\frac{d}{dt}\Psi^\alpha(t) = \frac{1}{2E}\tilde{W}^\dagger diag(\tilde{m}_i^2)\tilde{W}\Psi^\alpha(t) \ , \tag{5.44}$$

with

$$\Psi^\alpha(t) = \begin{pmatrix} \langle\nu_1|\tilde{\nu}_\alpha(t)\rangle \\ \langle\nu_2|\tilde{\nu}_\alpha(t)\rangle \\ \langle\nu_3|\tilde{\nu}_\alpha(t)\rangle \end{pmatrix} \ . \tag{5.45}$$

To solve this, the initial condition of Eq. (5.38) can be used to give

$$\Psi_k^\alpha(t) = \sum_i (\tilde{W}^\dagger)_{ki} e^{-i\frac{\tilde{m}_i^2}{2E}t}(\tilde{W}\tilde{\mathcal{U}}^\dagger)_{i\alpha} \ . \tag{5.46}$$

Now the amplitude for a transition from initial state $\tilde{\nu}_\alpha$ to a final state $\tilde{\nu}_\beta$ over a baseline L can be calculated as

$$A_{\alpha\to\beta} = \langle\tilde{\nu}_\beta(0)|\tilde{\nu}_\alpha(L=t)\rangle = \sum_{i=1}^{3}\overline{W}_{\beta i}\overline{W}_{\alpha i}^* e^{-i\frac{\tilde{m}_i^2}{2E}L} \ . \tag{5.47}$$

Here $\overline{W}$ denotes the non-unitary mixing matrix in matter

$$\overline{W}_{\alpha i} = (\tilde{\mathcal{U}}\tilde{W}^\dagger)_{\alpha i} = \lambda_\alpha^{-1} \sum_k \mathcal{U}_{\alpha k} \tilde{W}^*_{ik} \equiv \lambda_\alpha^{-1} W_{\alpha i} \ . \tag{5.48}$$

The probability amplitude is then

$$\begin{aligned} P_{\alpha\to\beta} \quad = \quad & |A_{\alpha\to\beta}|^2 \\ & \frac{1}{\lambda_\alpha^2\lambda_\beta^2}\Bigg\{ \left(\delta_{\alpha\beta} - \left|(\mathcal{V}\mathcal{V}^\dagger)_{\alpha\beta}\right|\right)^2 \\ & -4\sum_{i>k} \mathcal{R}e\left[W_{\alpha i}W_{\beta k}W^*_{\alpha k}W^*_{\beta i}\right]\sin^2\Delta_{ik} \\ & +8\mathcal{I}m\left[W_{\alpha 1}W_{\beta 2}W^*_{\alpha 2}W^*_{\beta 1}\right]\sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32} \\ & +2\left[\mathcal{I}m\left[W^*_{\alpha 1}W_{\beta 1}c^*_{\alpha\beta}\right]\sin 2\Delta_{31} + \mathcal{I}m\left[W^*_{\alpha 2}W_{\beta 2}c^*_{\alpha\beta}\right]\sin 2\Delta_{32}\right]\Bigg\} \ , \end{aligned} \tag{5.49}$$

where

$$\Delta_{ik} = 1.267\frac{(\tilde{m}_i^2 - \tilde{m}_k^2)[eV^2]L[km]}{E[GeV]} \ . \tag{5.50}$$

Chapter 6

Discussion

6.1 Dissertation review

In Chapter 1 the basic framework of known neutrino physics within the standard model (SM) was introduced and some possible extensions due to different mass models were also mentioned. These mass models corresponded to neutrinos having either Dirac mass or Majorana mass, with the see-saw mechanism also being introduced to solve the problem of why right-handed neutrinos have never been observed in experiments. Chapter 2 then introduced the concept of neutrino oscillations and the differences between neutrinos propagating in a vacuum and those propagating in matter. This then required us to take a detailed look at what experimental evidence actually existed for the oscillation phenomenon and which Chapter 3 was mainly focused on.

However toward the end of Chapter 3 it was clear that a new kind of experiment was needed in order to make any significant step forward in the determination of the parameters of the neutrino mixing matrix. On top of this, there were other important questions that needed answered, such as whether CP violation occurs in the leptonic sector of the SM. This lead us to the introduction of the concept of a neutrino factory.

In looking at the many advantages that a neutrino factory would have over a conventional beam experiment it was evident that the effects from new physics (NP) could be observed, due to the high energies that would be attainable and the flexibility of varying the baselines to detectors all over the world.

Chapters 4 and 5 went into more detail on how these new physics effects could be manifested through Non-Standard Neutrino Interactions (NSNI) and Non-Decoupling of Heavy Neutrinos (NDHN). It was found that both NSNI and NDHN relied on the neutrino states produced/detected as being in some other basis from that of the weak interaction basis.

6.2 Distinguishing Non-Standard Neutrino Interactions from Non-Unitarity

There is however an important difference between these two theories, that is, NDHN corresponds to the more fundamental theory of neutrino mixing, where the light neutrino mixing matrix is non-unitary. In the case of NSNI, the non-unitarity of the neutrino mixing matrix is only an extension to allow the incorporation of NP into the observation probabilities of the different neutrino flavours. The actual light neutrino mixing matrix does not become non-unitary. It is necessary to expand further on this point.

For NSNI a source state electron-neutrino $|\nu_e^s\rangle$ corresponding to the so-called "real" neutrino state produced in an experiment is described as

$$|\nu_e^s\rangle = \sum_i^3 U_{ei}^s |\nu_i^m\rangle \ . \tag{6.1}$$

This leaves the mixing matrix $U_{\alpha i}^s$ as non-unitary because $U_{ei}^s = [U_{ei}^W + \epsilon_{e\mu}^s U_{\mu i}^W + \epsilon_{e\tau}^s U_{\tau i}^W]$ and similarly for $U_{\mu i}^s$ and $U_{\tau i}^s$, where $U_{\alpha i}^W$ is the unitary light neutrino mixing matrix for the weak interaction eigenstates. It is only when one looks at the effective theory of four-fermion vertices that it becomes apparent this approach is just a convenient extension.

The part of the four-fermion vertex of interest for NSNI is

$$\begin{aligned}\sum_{i=1}^3 \ell_\alpha G_F \underbrace{U_{\alpha i}^s \nu_i^m}_{\nu_\alpha^s} &= \sum_{i=1}^3 \ell_\alpha G_F U_{\alpha i}^W \nu_i^m + \sum_{i=1}^3 \ell_\alpha \underbrace{G_F \epsilon_{\alpha\beta}^s}_{(G_{NP})_{\alpha\beta}^s} U_{\beta i}^W \nu_i^m \\ &= \sum_{i=1}^3 \ell_\alpha (G_{\alpha\beta})^{NSNI} U_{\beta i}^W \nu_i^m \end{aligned} \tag{6.2}$$

where $(G_{\alpha\beta})^{NSNI}$ is a non-diagonal matrix of couplings, comprising of a diagonal matrix $G_F\delta_{\alpha\beta}$ and a non-diagonal matrix $G_F\epsilon_{\alpha\beta}^s = (G_{NP})_{\alpha\beta}^s$. Here it is easy to see that the light neutrino mixing matrix has not actually become non-unitary and that it is the non-diagonal coupling $(G_{\alpha\beta})^{NSNI}$ which causes flavour changing interactions.

For NDHN an electron-neutrino state $|\tilde{\nu}_e\rangle$ corresponding to the so-called "real" neutrino state produced in an experiment is described as

$$|\tilde{\nu}_e\rangle = \sum_i^3 \tilde{\mathcal{U}}_{ei} |\nu_i^m\rangle \ , \tag{6.3}$$

where it is assumed no light sterile neutrinos are present, only heavy neutrinos. The mixing matrix $\tilde{\mathcal{U}}_{\alpha i}$ is also non-unitary because $\tilde{\mathcal{U}}_{\alpha i} = \lambda_\alpha^{-1}\mathcal{U}_{\alpha i}$ with $\lambda_\alpha = \sqrt{1 - c_{\alpha\alpha}}$ and $\mathcal{U}_{\alpha i}$ is a non-unitary matrix.

The part of the four-fermion vertex of interest for NDHN is

$$\sum_{i=1}^{n_R} \ell_\alpha G_F (U_\nu)_{\alpha i} \nu_i^m = \sum_{i=1}^{3} \ell_\alpha G_F \tilde{\mathcal{U}}_{\alpha i} \nu_i^m + \underbrace{\sum_{i=4}^{3+n_R} \ell_\alpha G_F \tilde{\mathcal{V}}_{\alpha i} \nu_i^m + \cdots}_{\text{assume negligable}} , \tag{6.4}$$

and it can be seen that it is the non-unitary mixing matrix that now causes flavour changing interactions. However the non-unitary matrix $\tilde{\mathcal{U}}_{\alpha i}$ can always be made from the unitary light neutrino mixing matrix $U_{\alpha i}^W$ and another matrix X, as in Eq. (5.40)

$$\tilde{\mathcal{U}}_{\alpha i} = X_{\alpha\beta} U_{\beta i}^W = \delta_{\alpha\beta} U_{\beta i}^W + \epsilon_{\alpha\beta} U_{\beta i}^W \ . \tag{6.5}$$

When finding probabilities at neutrino oscillation experiments, it is the mixing matrix that is used in calculations. In both NSNI Eq. (6.1) and Non-Unitarity (from NDHN) Eq. (6.3), the mixing matrices can be decomposed into a unitary part and a non-unitary part. So there is a problem in distinguishing NSNI from Non-Unitarity, that can only be solved if the ϵ parameters for each were theoretically determined to be significantly different.

For Non-Unitarity, the ϵ parameters should be the same for all experiments, regardless of what particles are involved, as they come from inside the full neutrino mixing matrix[1]. For NSNI the ϵ parameters could be at different scales, due to perhaps slepton masses being lighter than squark masses (see section 4.6.2). Using a neutrino factory and an upgraded conventional beam this difference could be determined, as different interactions with quarks and leptons would be involved. It could of course be, that it is necessary to look outside neutrino experiments altogether, so that this problem of ϵ parameters is bypassed.

At the moment the prospect of having a built neutrino factory within the next ten to fifteen years to investigate this and other aspects of neutrino physics is quite promising. This would not only allow us to confirm our current theories within neutrino physics, but also gain some insight into whether our supersymmetric and grand unified theories are the right description of physics beyond the standard model.

[1]They are a result of the constraints imposed on the light neutrino mixing matrix from the full neutrino mixing matrix.

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Appendix A

Bi-unitary Transformation of Charged Lepton Mass Matrix

A well known theorem in matrix algebra is that if a matrix A is normal (commutes with its hermitian conjugate), then there exists a unitary matrix U such that $UAU^\dagger$ is diagonal. However the mass matrices in a gauge theory do not have to be normal and even if they are normal there is no guarantee that when they are diagonalised using the above technique that the elements will be non-negative. This poses a problem for diagonalising the charged lepton mass matrix, as the diagonal elements in a mass matrix must be non-negative always, so that they can be interpreted as the masses of physical particles. Another theorem is therefore required:

Theorem: For any matrix A, one can find two unitary matrices U and V such that $VAU^\dagger$ is diagonal with non-negative elements.

Proof: It is clear that $AA^\dagger$ will be hermitian for any matrix A. Let

$$H^2 = AA^\dagger \ , \tag{A.1}$$

which may be diagonalised using the unitary matrix V:

$$VAA^\dagger V^\dagger = D^2 \ . \tag{A.2}$$

It should be noted that the elements $(D^2)_{ii} = \sum_j |(VA)_{ij}|^2$ and are therefore all real and non-negative. Let D be a diagonal matrix whose elements are the positive square roots of D^2, one then has $H \equiv V^\dagger DV$ as a hermitian matrix that satisfies (A.1).

It is clear that $H^{-1}A = V'$ will be a unitary matrix. One then has $A = HV' = V^\dagger DU$, where $U = VV'$ is a unitary matrix. Therefore $VAU^\dagger = D$ where D is the required diagonal matrix with non-negative elements. This result is valid even for singular matrices (ie. if A is not invertible).□

Mass terms can be introduced to $\mathcal{L}_{SM}$ for charged leptons in the form of Dirac mass terms

$$\mathcal{L}_{\rm mass} = -m\bar{\ell}\ell = -m(\overline{\ell}_L\ell_R + \overline{\ell}_R\ell_L) \ . \tag{A.3}$$

Taking the first term as an example, it can in general (ie. for all charged leptons) be denoted by

$$-\overline{\ell}_{iL}M_{ij}\ell_{jR} \ . \tag{A.4}$$

If the mass matrix M is not diagonal then there will be mixing between the charged leptons. Using the above theorem one can diagonalise M,

$$-\underbrace{\overline{\ell}_{kR}V^{\ell}_{ki}}_{\overline{\tilde{\ell}}_{iR}}M_{ij}\underbrace{(U^{\ell})^{\dagger}_{jm}\ell_{mL}}_{\tilde{\ell}_{jL}} \ . \tag{A.5}$$

Or alternatively the charged lepton fields ℓ_i can be seen as rotations of charged lepton mass fields $\tilde{\ell}_i$,

$$\ell_{\rm L} = U^{\ell}\tilde{\ell}_{\rm L} \ , \qquad \ell_{\rm R} = V^{\ell}\tilde{\ell}_{\rm R} \ . \tag{A.6}$$

Appendix B

Representations of the Lorentz Group

This appendix is based on material in an appendix on spinors in an article written by R. D. Peccei on neutrino physics [10].

B.1 Weyl spinors

The Lorentz group which has vector and tensor representations, can also have spinor representations. Spinors, which transform under these spinor representations of the Lorentz group will be the main concern of this Appendix.

The Lorentz group $SO(3,1)$ is defined by the algebra

$$SO(3,1) = \{\Lambda \in R^{4\times 4} : \Lambda^T G \Lambda = G,\ G = \text{diag}(-1,\ 1,\ 1,\ 1) \text{ and } \det\Lambda = 1\}\ . \tag{B.1}$$

This algebra is isomorphic to the algebra of $SL(2,\mathbb{C})$, the group of 2×2 complex matrices of determinant one. There are two inequivalent representations of this group which shall be introduced as 2×2 matrices M and M^*. The corresponding 2-dimensional spinors that transform under these representations are the Weyl spinors ξ_a and $\dot{\xi}_a$ respectively $a = \{1,2\}$. The relevant transformations are then

$$\xi_a \to \xi'_a = M_a{}^b \xi_b \qquad (\frac{1}{2}, 0) \tag{B.2}$$

$$\dot{\xi}_a \to \dot{\xi}'_a = M_a^{*b} \dot{\xi}_b \qquad (0, \frac{1}{2})\ . \tag{B.3}$$

The algebra of the Lorentz group is also isomorphic on $\mathbb{C}$ to the algebra of the semi-simple group $SU(2)^+ \times SU(2)^-$. The finite-dimensional irreducible representations of $SL(2,\mathbb{C})$ can therefore be classified by the non-negative integer or half integer indicies (ℓ_+, ℓ_-) of the finite-dimensional irreducible representations of $SU(2)^+ \times SU(2)^-$. It is evident from Eqs. (B.2) and (B.3) that $\dot{\xi} \backsim \xi^*$, in other words they transform in the same way up to some phase factor.

The spinor analogues of the scalar product for vectors $[g_{\mu\nu}V^\mu V^\nu \equiv V^\mu V_\mu]$ are

$$\epsilon^{ab}\xi_a\xi_b \equiv \xi_a\xi^b \; ; \qquad \epsilon^{ab}\dot{\xi}_a\dot{\xi}_b \equiv \dot{\xi}_a\dot{\xi}^b \tag{B.4}$$

where $\epsilon^{ab} = -\epsilon^{ba}$ and $\epsilon^{12} = 1$. For vectors the contraction of a covariant and contravariant metric tensor gives the identity $g_{\mu\rho}g^{\rho\nu} = \delta^\nu_\mu$. In a similar fashion one can define 2×2 antisymmetric ϵ-matrices ϵ_{ab}, which obey the relation

$$\epsilon_{ac}\epsilon^{cb} = \delta^b_a \; . \tag{B.5}$$

Therefore $\epsilon_{12} = -1$ and one can use these ϵ-matrices to raise and lower spinor indices in the same way as the metric tensor does for vectors.

B.2 Dirac spinors

A 4-component Dirac spinor ψ_D is made out of two Weyl spinors, one dotted the other undotted:

$$\psi_D = \begin{pmatrix} \xi_a \\ \dot{\chi}^a \end{pmatrix} \; . \tag{B.6}$$

This basis for the Dirac spinor is known as the Weyl basis. The Dirac γ-matrices γ^μ from the Dirac equation obey the basis independent anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$. In the Weyl basis they take the matrix form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \; ; \qquad \text{with } \sigma^\mu = (1, \vec{\sigma}), \; \bar{\sigma}^\mu = (1, -\vec{\sigma}) \; . \tag{B.7}$$

Separately this gives

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \; ; \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \; . \tag{B.8}$$

The chiral projection operators P_L and P_R are defined as

$$P_L = \frac{1}{2}(1 - \gamma_5) \; ; \qquad P_R = \frac{1}{2}(1 + \gamma_5) \; . \tag{B.9}$$

It is clear that when using this basis for the γ-matrices, the Weyl spinors ξ_a and $\dot{\chi}^a$ are actually chiral projections of the Dirac spinor ψ_D:

$$(\psi_D)_\mathrm{L} = P_L\psi_D = \begin{pmatrix} \xi_a \\ 0 \end{pmatrix} \; ; \quad \overline{(\psi_D)_\mathrm{L}} = \overline{\psi_D}P_R = (0 \;\; \xi^*_a) \tag{B.10}$$

$$(\psi_D)_\mathrm{R} = P_R\psi_D = \begin{pmatrix} 0 \\ \dot{\chi}^a \end{pmatrix} \; ; \quad \overline{(\psi_D)_\mathrm{R}} = \overline{\psi_D}P_L = (\dot{\chi}^{a*} \;\; 0) \; . \tag{B.11}$$

Here $\overline{\psi_D} = \psi_D^\dagger \gamma^0$ denotes the conjugate spinor field. In the ultrarelativistic limit $E/m \to \infty$, the helicity (spin projection on the direction of motion or handedness) of a particle is directly related to its chirality (eigenstates of γ_5). In this limit the helicity operator $\vec{\Sigma} \cdot \vec{p}$ with eigenvalues $\lambda = \pm\frac{1}{2}$ (where the corresponding eigenstates are known as left- and right-handed) becomes equivalent to γ_5. Therefore the chiral projections as defined in Eqs. (B.10) and (B.11) are loosely known as left- and right-handed.

A lorentz invariant mass term for a Dirac spinor takes the form

$$\mathcal{L}_{\text{Dirac}} = -m_D \overline{\psi_D} \psi_D = -m_D(\overline{(\psi_D)_{\text{L}}}(\psi_D)_{\text{R}} + \overline{(\psi_D)_{\text{R}}}(\psi_D)_{\text{L}}) \ , \tag{B.12}$$

where the other terms cancel due to the projection operators. Using Eqs. (B.10) and (B.11) one finds that the two Weyl spinors are connected by the mass term

$$\mathcal{L}_{\text{Dirac}} = -m_D(\xi_a^* \dot{\chi}^a + \dot{\chi}^{a*} \xi_a) \ . \tag{B.13}$$

From appendix B.1 one should recall that a dotted Weyl spinor transforms in the same way as the complex conjugate of undotted Weyl spinor. Therefore by choosing an appropriate phase convention

$$\xi_a^* = \dot{\xi}_a \ ; \qquad \dot{\chi}^{a*} = \chi^a, \tag{B.14}$$

the Dirac mass term can then be written as

$$\mathcal{L}_{\text{Dirac}} = -m_D(\dot{\xi}_a \dot{\chi}^a + \chi^a \xi_a) \ . \tag{B.15}$$

B.3 Majorana spinors

A 4-component Majorana spinor ψ_M is made out of a Weyl spinor ξ and its complex conjugate $\dot{\xi}$:

$$\psi_M = \begin{pmatrix} \xi_a \\ \dot{\xi}^a \end{pmatrix} \ . \tag{B.16}$$

Again by choosing an appropriate phase convention one has $\dot{\xi}^a = \xi^{a*}$. This means that ψ_M has only one independent chiral projection which can be chosen as $(\psi_M)_{\text{L}}$:

$$(\psi_M)_{\text{L}} = P_L \psi_M = \begin{pmatrix} \xi_a \\ 0 \end{pmatrix} \ ; \qquad \overline{(\psi_M)_{\text{L}}} = \overline{\psi_M} P_R = (0 \ \ \dot{\xi}_a) \ , \tag{B.17}$$

$(\psi_M)_{\text{R}}$ can then be made by using the charge conjugate matrix $\tilde{C}$. In the Weyl basis $\tilde{C}$ is given by the matrix

$$\tilde{C} = \begin{pmatrix} \epsilon_{ab} & 0 \\ 0 & \epsilon^{ab} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ . \tag{B.18}$$

One then has

$$
\begin{aligned}
(\psi_M)_{\mathrm{R}} &= \begin{pmatrix} 0 \\ \dot{\xi}^a \end{pmatrix} = \begin{pmatrix} 0 \\ \epsilon^{ab}\dot{\xi}_b \end{pmatrix} = \tilde{C}\overline{(\psi_M)_{\mathrm{L}}}^T \\
\overline{(\psi_M)_{\mathrm{R}}} &= (\xi^a \ 0) = (\epsilon^{ab}\xi_b \ 0) = (\xi_b\epsilon^{ba} \ 0) = (\psi_M)_{\mathrm{L}}^T\tilde{C} \ .
\end{aligned} \tag{B.19}
$$

The charge conjugate matrix is introduced to insure the invariance of the Dirac equation under charge conjugation. For a spinor field the Dirac equation reads

$$[\gamma^\mu(i\partial_\mu + eA_\mu) - m]\psi = 0 \ , \tag{B.20}$$

and for the charge conjugate field ψ_c it reads

$$[\gamma^\mu(i\partial_\mu - eA_\mu) - m]\psi_c = 0 \ . \tag{B.21}$$

It is therefore necessary to insure invariance by defining $\psi_c = \tilde{C}\overline{\psi}^T$ with the following restrictions on $\tilde{C}$

$$\tilde{C}\gamma_\mu^T\tilde{C}^{-1} = -\gamma_\mu, \qquad C = C^{-1}, \qquad C^T = -C \ . \tag{B.22}$$

One then finds in the Weyl basis of the γ matrices that $\overline{\psi}_c = \psi^T\tilde{C}$. Therefore from Eq. (B.19), $(\psi_M)_{\mathrm{R}}$ is the charge conjugate of $(\psi_M)_{\mathrm{L}}$

$$[(\psi_M)_{\mathrm{L}}]^c = (\psi_M)_{\mathrm{R}} \ . \tag{B.23}$$

It is now also evident that ψ_M is self-conjugate

$$\psi_M = \begin{pmatrix} \xi_a \\ \dot{\xi}^a \end{pmatrix} = (\psi_M)_{\mathrm{L}} + (\psi_M)_{\mathrm{R}} = (\psi_M)_{\mathrm{L}} + [(\psi_M)_{\mathrm{L}}]^c = [\psi_M]^c \ . \tag{B.24}$$

This concept where a particle is identical to its antiparticle was introduced by Majorana in 1937 [13]. A Lorentz invariant mass term for a Majorana spinor takes the same form as that of the Dirac spinor

$$\mathcal{L}_{\mathrm{Majorana}} = -\frac{1}{2}m_M\overline{\psi_M}\psi_M = -\frac{1}{2}m_M\left(\overline{(\psi_M)_{\mathrm{L}}}(\psi_M)_{\mathrm{R}} + \overline{(\psi_M)_{\mathrm{R}}}(\psi_M)_{\mathrm{L}}\right) \ , \tag{B.25}$$

where the other terms cancel due to the projection operators. By substituting Eqs. (B.17) and (B.19) into (B.25) one finds that the Weyl spinor and its complex conjugate are connected by the mass term

$$\mathcal{L}_{\mathrm{Majorana}} = -\frac{1}{2}m_M(\dot{\xi}_a\dot{\xi}^a + \xi^a\xi_a) \ . \tag{B.26}$$

Which again is Lorentz invariant. Using Eq. (B.19) the mass term of Eq. (B.25) can also be written purely in terms of either $(\psi_M)_{\mathrm{L}}$ or $(\psi_M)_{\mathrm{R}}$ using the charge conjugation matrix $\tilde{C}$:

$$\mathcal{L}_{\mathrm{Majorana}} = -\frac{1}{2}m_M\left(\overline{(\psi_M)_{\mathrm{L}}}\tilde{C}\overline{(\psi_M)_{\mathrm{L}}}^{\ T} + (\psi_M)_{\mathrm{L}}^T\tilde{C}(\psi_M)_{\mathrm{L}}\right) \ \text{or} \tag{B.27}$$

$$\mathcal{L}_{\mathrm{Majorana}} = -\frac{1}{2}m_M\left((\psi_M)_{\mathrm{R}}^T\tilde{C}(\psi_M)_{\mathrm{R}} + \overline{(\psi_M)_{\mathrm{R}}}\tilde{C}\overline{(\psi_M)_{\mathrm{R}}}^{\ T}\right) \ . \tag{B.28}$$